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MAPPING CONTINUA ONTO THEIR CONES

BY

DAVID P. BELLAMY (NEWARK, DELAWARE)
AND CHARLES L. HAGOPIAN (SACRAMENTO, CALIFORNIA)

The question was raised several years ago at one of the Annual Spring Topology Conferences in the United States as to whether there exists a compact metric continuum which cannot be mapped onto the cone over itself. The question was attributed by Howard Cook to the first-named author, in connection with the results in [1], although he has no memory of having asked it. This question is interesting since an example must be intermediate between "nice" continua and "bad" continua, and such continua are often difficult to study. Precisely, if X is a Peano continuum, then so is $C(X)$, the cone over X, and so each can be mapped onto the other by the Hahn-Mazurkiewicz theorem. On the other hand, if X contains an open set with uncountably many components, it can be mapped onto the cone over the Cantor set, and hence onto the cone over any compact metric space, itself included (see [1], Theorem II and Lemma I, p. 14 and 15).

A knowledge of the set-function T is assumed in one argument in the sequel. For its definition and basic properties see [2]-[5].

In [6], the second-named author introduces a class of continua called quasi-Peano. A continuum X is *quasi-Peano* if there exists a compact totally disconnected metric space D such that X is both a continuous image and a continuous preimage of $C(D)$.

For a given compactum X, denote by $[x, t]$ a typical point in $C(X)$, where $[x, 1]$ is the vertex for any $x \in X$. If I denotes the closed unit interval, then — referring again to the Hahn-Mazurkiewicz theorem — there is a map μ of I onto $I \times I$ such that $\mu(0) = \mu(1) = (1, 1)$. (The restriction that $\mu(0) = (1,1)$ will not be used in the following argument, but will be used in an example at the end of the paper.) Let $\mu(t) = (\mu_1(t), \mu_2(t))$.

THEOREM 1. *For any compactum X, $C(X)$ can be mapped continuously onto $C(X) \times I$, and hence onto $C(C(X))$.*

Proof. Define $g: C(X) \to C(X)\times I$ by

$$g[x,t] = \big([x,\mu_1(t)],\mu_2(t)\big).$$

Then

$$g(x,1) = \big([x,\mu_1(1)],\mu_2(1)\big) = \big([x,1],1\big),$$

and so g is well defined; g is onto since if $([x,s],t)\in C(X)\times I$, then there is a $p\in I$ such that $\mu(p) = (s,t)$, and so $g[x,p] = ([x,s],t)$. The continuity of g is clear since the following diagram commutes and q is an identification map:

$$\begin{tikzcd} X\times I \arrow[r, "1\times\mu"] \arrow[d, "q"'] & X\times I\times I \arrow[d, "q\times 1_I"] \\ C(X) \arrow[r, "g"] & C(X)\times I \end{tikzcd}$$

THEOREM 2. *Assume that each of X and Y can be mapped onto the other. Then X can be mapped onto $C(X)$ if and only if Y can be mapped onto $C(Y)$.*

Proof. By symmetry it suffices to prove only one assertion. Let $f: X\to Y$ and $g: Y\to X$ be continuous and onto. If $h: Y\to C(Y)$ is continuous and onto, then

$$C(g)\circ h\circ f: X\to C(X)$$

is continuous and onto, where $C(g): C(Y)\to C(X)$ is the map $C(g)[y,t] = [g(y),t]$.

Thus, by Theorems 1 and 2, if X is a quasi-Peano continuum, then X can be mapped onto $C(X)$. The second-named author gives an example of a non quasi-Peano continuum in [6], and this turns out to be an example of a continuum X which cannot be mapped onto $C(X)$. For a geometric construction and picture of this continuum, see [6]. A quick construction is given here.

Example 1. *A metric continuum X which cannot be mapped onto its cone.*

Let A denote the one-point compactification of $C(\omega+1)\times\omega$, where ω is the first infinite ordinal. Let v denote the vertex of $C(\omega+1)$. The continuum X is obtained from A by identifying (v,n) with $([n-1,0],0)$ for each $n\geqslant 1$ and the compactification point p with the point $([\omega,0],0)$.

Let $B\subseteq C(X)$ be the cone over the set

$$\{(v,n)\}_{n=0}^{\infty}\cup\{[\omega,0],0\} \qquad (\text{or } \{v,n\}_{n\in\omega}\cup\{p\}).$$

It is easily seen that $T(B)$ is the cone in $C(X)$ over $\bigcup_{i=0}^{\infty} L_i$, where L_i is the limit continuum in $C(\omega+1)\times\{i\}$. Observe that if $\langle M_i\rangle_{i=1}^{\infty}$ is a sequence of Peano continua in $C(X)$ such that $\lim_{i\to\infty} M_i$ is a single point, then $T(B)-B$ cannot be contained in $\bigcup_{i=1}^{\infty} M_i$. Indeed, if so, then

$$T(B) \subseteq \bigcup_{i=1}^{\infty} M_i \cup \{q\}, \quad \text{where } q = \lim_{i\to\infty} M_i,$$

since $\bigcup_{i=1}^{\infty} M_i \cup \{q\}$ is closed and $\mathrm{Cl}\big(T(B)-B\big) = T(B)$. However, $\bigcup_{i=1}^{\infty} M_i \cup \{q\}$ is locally connected except possibly at q, whereas any subset of $C(X)$ containing $T(B)$ must fail to be locally connected at all points of the form $[p, t]$ for $t < 1$.

To prove that there is no map of X onto $C(X)$, suppose that f is such a map. By Lemma 14 of [2], p. 587,

$$T_{C(X)}(B) \subseteq fT_X f^{-1}(B),$$

where the subscripts on T denote the continuum with respect to whose topology T is computed.

Now, observe that

$$T_X f^{-1}(B) \subseteq f^{-1}(B) \cup \big(\bigcup_{n=0}^{\infty} L_n\big)$$

(by, for example, the remarks in the second section of the introduction of [4]), so that

$$fT_X f^{-1}(B) \subseteq B \cup f\big(\bigcup_{n=0}^{\infty} L_n\big) \quad \text{and} \quad T_{C(X)}(B) \subseteq B \cup \big(\bigcup_{n=0}^{\infty} f(L_n)\big).$$

However, $\{f(L_n)\}_{n=0}^{\infty}$ is a sequence of Peano continua converging to a point $f(p)$, and by the observation above,

$$T(B)-B \nsubseteq \bigcup_{n=0}^{\infty} f(L_n),$$

a contradiction.

Finally, there are non-quasi Peano continua which can be mapped onto their cones; the argument which follows shows that the usual curve $\sin x^{-1}$ can be so mapped. The curve $\sin x^{-1}$ can be obtained from $(\omega+1)\times I$ by identifying $(n, 1)$ with $(n+1, 1)$ for n odd and $(n, 0)$ with $(n+1, 0)$ for n even. Denote the identification map by η, let q: $S \times I \to C(S)$ be the usual quotient map, and let μ: $I \to I \times I$ be the map introduced above. It is clear that if $\eta(x) = \eta(y)$, then

$$q \circ (\eta \times 1_I) \circ (1_{\omega+1} \times \mu)(x) = q \circ (\eta \times 1_I) \circ (1_{\omega+1} \times \mu)(y),$$

so that a continuous surjection g exists completing the following diagram:

$$\begin{tikzcd}
(\omega+1)\times I \arrow[r, "1_{\omega+1}\times\mu"] \arrow[dd, "\eta"'] & (\omega+1)\times I\times I \arrow[d, "\eta\times 1_I"] \\
 & S\times I \arrow[d, "q"] \\
S \arrow[r, dotted, "g"] & C(S)
\end{tikzcd}$$

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UNIVERSITY OF DELAWARE
NEWARK, DELAWARE

CALIFORNIA STATE UNIVERSITY
SACRAMENTO, CALIFORNIA

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