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OPTIMIZATION PROBLEMS ON COMPLETE RIEMANNIAN MANIFOLDS

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1. Introduction. In this paper we give some applications of the variational principle of Ekeland [5] to optimization problems on complete Riemannian manifolds. The optimization problems presented here are global versions of some problems already treated on Banach spaces [6]. They are motivated by specific problems on Riemannian manifolds such as the problem of the minimal geodesics joining two closed subsets.

In order to apply the variational principle of Ekeland we shall work only with complete Riemannian manifolds (in view of [15] the hypothesis of completeness is essential).

In Section 2 we start with the problem of the minimal geodesics with variable endpoints. First, for finite-dimensional manifolds we extend a result of Grove [8], and then in the infinite-dimensional case we obtain some approximate solutions to the problem of the minimal geodesics between two closed subsets. This gives a motivation to treat a general problem of mathematical programming on a complete Riemannian manifold. In Section 3, using the concept of generalized gradient (see [1], [2] and [10]), we give an estimation of the distance between two closed subsets of a complete Riemannian manifold. In Section 4 we state a fixed point theorem under the hypothesis of a contraction along a minimal geodesic (this is a generalization of a theorem of Clarke [3]).

2. Minimal geodesics and mathematical programming. Let M be a (possibly infinite-dimensional) Riemannian manifold which we further suppose to be complete and connected. Then each tangent space TM_p is endowed with an inner product $(\cdot, \cdot)_p$ depending smoothly on the point p and let $\|\cdot\|_p$ be the norm on TM_p induced by $(\cdot, \cdot)_p$. Denote by d the Riemannian distance on M.

In the setting of Riemannian manifolds we need to consider the notions of generalized gradient and normal cone. Recall that these notions are due to Clarke who introduced them in the case of Banach spaces (see [1] and [2]). Our approach to Riemannian manifolds is based on the use of the exponential map $\exp_p\colon TM_p \to M$.

DEFINITION 1. Let $f\colon M \to R$ be a locally Lipschitz function on M and let $p \in M$. The *generalized gradient* of f at p, denoted by $\partial f(p)$, is defined to be the generalized gradient $\partial(f \circ \exp_p)(O_p)$ (in the sense of Clarke) of the locally Lipschitz function $f \circ \exp_p$ on the Hilbert space TM_p at $O_p \in TM_p$.

DEFINITION 2. Let S be a closed subset of M and let $p \in S$. We define the *normal cone* $N_S(p)$ to S at p as the normal cone in the sense of Clarke $N_{\exp_p^{-1}(S)}(O_p)$ to the closed subset $\exp_p^{-1}(S)$ of the Hilbert space TM_p at $O_p \in TM_p$.

Remark 1. These notions as well as other concepts and results from optimization theory were generalized in [10] to Banach manifolds.

Now, let S_0 and S_1 be two disjoint and closed subsets of M. In the following we shall be concerned with the problem of the existence of minimal geodesics between S_0 and S_1. If M is finite dimensional and S_i $(i = 0, 1)$ are submanifolds of M, then it is well known that there exists a minimal geodesic between S_0 and S_1 which in addition is normal both to S_0 and S_1 (see [8], Theorem 2.6). First of all we shall briefly establish that we can drop the hypothesis that S_i $(i = 0, 1)$ are submanifolds.

THEOREM 1. *Let M be a finite-dimensional connected complete Riemannian manifold and let S_i $(i = 0, 1)$ be two disjoint and closed subsets of M provided at least one of them is compact. Then there exists a curve $c\colon [0, 1] \to M$ from S_0 to S_1 such that*

(i) *c is a geodesic of minimal length between S_0 and S_1, i.e.,*

$$L(c) = d(S_0, S_1) = \inf_{x_i \in S_i} d(x_0, x_1);$$

(ii) *$c'(0) \in N_{S_0}(c(0))$ and $c'(1) \in -N_{S_1}(c(1))$.*

Proof. Assume, e.g., that S_1 is compact. Putting $a = d(S_0, S_1)$, we see that

$$S_0' = \{x \in S_0;\ d(x, S_1) \leqslant a+1\}$$

is a nonempty compact subset of M and

$$d(S_0', S_1) = d(S_0, S_1) = a.$$

Hence there exist points $p \in S_0$ and $q \in S_1$ such that $d(p, q) = a$. The Hopf-Rinow Theorem assures the existence of a minimal geodesic c joining p and q. Thus $L(c) = d(p, q) = a$, which establishes (i).

Concerning (ii) we shall prove only that $c'(1) \in -N_{S_1}(c(1))$, the same argument yielding the other relation. Let d_p be the distance function $d_p(x)$

$= d(p, x)$. In view of the above argument and with the notation of (i) it is clear that the point $O_q \in TM_q$ minimizes the function $d_p \circ \exp_q$ over $\exp_q^{-1}(S_1)$. Then using Lemma 2 from [2] and applying the calculus with generalized gradients (see [1] and [2]), we obtain

$$O_q \in \operatorname{grad}(d_p \circ \exp_q)(O_q) + N_{\exp_q^{-1}(S_1)}(O_q),$$

i.e.,

$$\operatorname{grad}(d_p)(q) \in -N_{S_1}(q).$$

But $c'(1) = a \operatorname{grad}(d_p)(q)$. This assertion can be easily checked by using, e.g., an isometric embedding into some Euclidean space (which exists by the Nash Embedding Theorem [11]). Then $c'(1)$ belongs to $-N_{S_1}(q)$, so the proof is complete.

For infinite-dimensional manifolds it is well known that even if S_0 and S_1 reduce to points, the minimal geodesics may not exist (see [6] for a counter-example and [7] for the infinite-dimensional version of the Hopf–Rinow Theorem). Hence in the infinite-dimensional case the question is in fact to approximate the minimal geodesics. The construction of "almost" minimal geodesics joining two given points was performed by Ekeland (see [5], Theorem 6.3). So it is natural to look for a result of this type when the points are replaced by closed subsets.

In the following, M denotes an infinite-dimensional complete and connected Riemannian manifold. Assume in addition that M is separable and is modelled by a separable Hilbert space. Consider the Sobolev Riemannian manifold $L_1^2([0, 1]; M)$ and recall that the tangent space $T(L_1^2([0, 1]; M))_c$ at $c \in L_1^2([0, 1]; M)$ is the set of all L_1^2 vector fields of M along c (see, e.g., [5] and [8]).

Let us consider the energy function $F\colon L_1^2([0, 1]; M) \to R$ given by

$$F(c) = \tfrac{1}{2} \int_0^1 \|c'(t)\|_{c(t)}^2 \, dt.$$

For the use of the energy integral in the theory of geodesics we refer to [5], [8], [12], and [13].

The problem of minimal geodesics between two fixed nonempty disjoint and closed subsets S_0 and S_1 of M is equivalent to the following optimization problem:

$$\text{(P)} \qquad \inf\{F(c)\colon c(0) \in S_0,\ c(1) \in S_1\}.$$

We shall proceed by reducing the problem (P) to a mathematical programming problem on M.

By [4], there exist positive smooth functions $g_0\colon M \to R$ and $g_1\colon M \to R$ corresponding to S_0 and S_1 such that

$$(1) \qquad S_i = g_i^{-1}(0), \quad i = 0, 1.$$

Write $G_i = g_i \circ \mathrm{ev}_i$ $(i = 0, 1)$, where ev_i: $L_1^2([0, 1]; M) \to M$ $(i = 0, 1)$ represent the evaluation mappings $\mathrm{ev}_0(c) = c(0)$ and $\mathrm{ev}_1(c) = c(1)$, respectively. From (1) it follows that (P) is equivalent to the following problem:

$$(\mathrm{P}') \qquad \inf\{F(c)\colon\ G_i(c) = 0,\ i = 0, 1\}.$$

We add the following natural hypothesis of regularity:

(2) both functions g_0 and g_1 have 0 as a regular value.

Remark 2. The assumption (2) implies that S_0 and S_1 are smooth submanifolds of M.

The following result describes some approximate smooth solutions of the problem (P′) (or (P)) which, in view of their properties, may be called "*almost*" *minimal geodesics* between S_0 and S_1.

THEOREM 2. *Let two nonempty disjoint and closed submanifolds S_0 and S_1 of M be given such that* (1) *and* (2) *hold. Then for each $\varepsilon > 0$ there exists a path $c_\varepsilon \in C^\infty([0, 1]; M)$ satisfying the following conditions*:

(i) $c_\varepsilon(0) \in S_0$, $c_\varepsilon(1) \in S_1$, *and*

$$F(c_\varepsilon) \leqslant \inf\{F(c)\colon\ c \in L_1^2([0, 1]; M),\ c(0) \in S_0,\ c(1) \in S_1\} + \varepsilon;$$

(ii) *there exist real numbers λ and μ such that*

$$\Big\|\int_0^1 (\nabla_{c_\varepsilon}(\cdot),\ c'_\varepsilon(t))_{c_\varepsilon(t)}\,dt - \lambda D(g_0)_{c_\varepsilon(0)} \circ \mathrm{Ev}_0 - \mu D(g_1)_{c_\varepsilon(1)} \circ \mathrm{Ev}_1\Big\|^* \leqslant \varepsilon,$$

where ∇_{c_ε} is the covariant derivative, $\|\cdot\|^*$ is the dual norm on $T(L_1^2([0, 1]; M))_{c_\varepsilon}^*$, and Ev_i: $T(L_1^2([0, 1]; M))_{c_\varepsilon} \to TM_{c_\varepsilon(i)}$ *represents the evaluation mappings* $\mathrm{Ev}_i(\xi) = \xi(i)$ $(i = 0, 1)$.

COROLLARY 1. *If we set*

$$V = \{\xi \in T(L_1^2([0, 1]; M))_{c_\varepsilon}\colon\ \xi(0) \in T(S_0)_{c_\varepsilon(0)} \text{ and } \xi(1) \in T(S_1)_{c_\varepsilon(1)}\},$$

then

$$(3) \quad \Big|\int_0^1 (\nabla_{c_\varepsilon}(\xi(t)),\ c'_\varepsilon(t))_{c_\varepsilon(t)}\,dt\Big|^2 \leqslant \varepsilon \int_0^1 (\|\xi(t)\|^2_{c_\varepsilon(t)} + \|\nabla_{c_\varepsilon}\xi(t)\|^2_{c_\varepsilon(t)})\,dt \qquad \text{for all } \xi \in V.$$

Moreover, if $c'_\varepsilon \in V$,

$$(4) \qquad \big|\|c'_\varepsilon(1)\|^2_{c_\varepsilon(1)} - \|c'_\varepsilon(0)\|^2_{c_\varepsilon(0)}\big| \leqslant 2\varepsilon \int_0^1 (\|c'_\varepsilon(t)\|^2_{c_\varepsilon(t)} + \|\nabla_{c_\varepsilon} c'_\varepsilon(t)\|^2_{c_\varepsilon(t)})\,dt.$$

Proof of Corollary 1. By (1) and (2) we have

$$T(S_0)_{c_\varepsilon(0)} = (D(g_0)_{c_\varepsilon(0)})^{-1}(0) \quad \text{and} \quad T(S_1)_{c_\varepsilon(1)} = (D(g_1)_{c_\varepsilon(1)})^{-1}(0).$$

Relation (3) is now obtained from (ii) of Theorem 2 and the definition of V.

Putting $\xi = c'_\varepsilon$ in (3) and using

$$\frac{d}{dt}\|c'(t)\|^2_{c_\varepsilon(t)} = 2\big(\nabla_{c_\varepsilon} c'_\varepsilon(t),\, c'_\varepsilon(t)\big)_{c_\varepsilon(t)},$$

we obtain (4).

Remark 3. Under Grove's notation [8] we have

$$V = T\big(\Lambda_{S_0 \times S_1}(M)\big)_{c_\varepsilon}.$$

We shall deduce Theorem 2 from another result (Theorem 3) by discussing a general minimization problem on a Riemannian manifold subject to equality and inequality constraints.

On a complete (possibly infinite-dimensional) Riemannian manifold M consider the following problem with constraints:

(5) $$\inf\{F(p)\colon G_i(p) = 0,\ 1 \leqslant i \leqslant h;\ G_i(p) \geqslant 0,\ h+1 \leqslant i \leqslant k\},$$

where $F\colon M \to R$ is a Fréchet differentiable function and $G_i\colon M \to R$ $(1 \leqslant i \leqslant k)$ are smooth functions on M.

This problem was studied by Ekeland in the case of Banach spaces [5]. Our next theorem is a version of Ekeland's Theorem for complete Riemannian manifolds. The main point in the proof is the construction of an appropriate geodesic.

Let

$$C = \{p \in M;\ G_i(p) = 0\ (1 \leqslant i \leqslant h) \text{ and } G_i(p) \geqslant 0\ (h+1 \leqslant i \leqslant k)\}.$$

For each $p \in C$ put

$$I(p) = \{i \in \{1, \ldots, k\}\colon G_i(p) = 0\}$$

and let $i(p)$ be the number of elements of $I(p)$.

THEOREM 3. *Assume that $F\colon M \to R$ is bounded from below and that for all $p \in C$ the following condition is satisfied:*

(6) $\{D(G_i)_p\colon i \in I(p)\}$ *are linearly independent in* TM^*_p.

Then for each $\varepsilon > 0$ there exists $p_\varepsilon \in C$ with the properties

(i) $F(p_\varepsilon) \leqslant \inf\limits_C F + \varepsilon$;

(ii) *there exist constants λ_i $(1 \leqslant i \leqslant k)$ such that*

$$\lambda_i \geqslant 0\ (h+1 \leqslant i \leqslant k), \quad \lambda_i G_i(p_\varepsilon) = 0\ (1 \leqslant i \leqslant k)$$

and

$$\Big\|D(F)_{p_\varepsilon} - \sum_{i=1}^{k} \lambda_i D(G_i)_{p_\varepsilon}\Big\|^*_{p_\varepsilon} \leqslant \varepsilon.$$

Proof. By the variational principle of Ekeland applied to the complete metric space C, there exists a point p_ε of C such that (i) is satisfied and

$$F(q) \geqslant F(p_\varepsilon) - \varepsilon d(p_\varepsilon, q) \quad \text{for all } q \in C. \tag{7}$$

Let $v \in TM_{p_\varepsilon}$ be such that

$$D(G_i)_{p_\varepsilon}(v) = 0 \quad (1 \leqslant i \leqslant h)$$

and

$$D(G_i)_{p_\varepsilon}(v) \geqslant 0 \quad (i \in \{h+1, \ldots, k\} \cap I(p_\varepsilon)).$$

Set

$$I(p_\varepsilon, v) = \{i \in \{1, \ldots, k\}: G_i(p_\varepsilon) = 0 \text{ and } D(G_i)_{p_\varepsilon}(v) = 0\}$$

and let $i(p_\varepsilon, v)$ be the number of elements of $I(p_\varepsilon, v)$.

Define the mapping $G: M \to R^{i(p_\varepsilon, v)}$ by

$$G(x) = (G_i(x))_{i \in I(p_\varepsilon, v)}.$$

Since, by (6), G is transversal to $\{0\} \subset R^{i(p_\varepsilon, v)}$, $G^{-1}(0)$ is a closed submanifold of M and

$$v \in T(G^{-1}(0))_{p_\varepsilon} = (D(G)_{p_\varepsilon})^{-1}(0).$$

We endow $G^{-1}(0)$ with the Riemannian structure induced by M. Let $c: [0, \infty) \to G^{-1}(0)$ be the unique geodesic of $G^{-1}(0)$ such that

$$c(0) = p_\varepsilon \quad \text{and} \quad c'(0) = v. \tag{8}$$

According to a standard property of the exponential mapping there is some $t_0 > 0$ such that

$$d(c(t), p_\varepsilon) = t\,\|v\|_{p_\varepsilon} \quad \text{for all } t \in [0, t_0]. \tag{9}$$

Now make the essential observation that we may choose $t_0 > 0$ with the additional property:

$$c(t) \in C \quad \text{for } t \in [0, t_0]. \tag{10}$$

Then (7), combined with (8)–(10), yields

$$\frac{d}{dt} F \circ c(t)\big|_{t=0} = D(F)_{p_\varepsilon}(v) \geqslant -\varepsilon\,\|v\|_{p_\varepsilon}. \tag{11}$$

Now (ii) is a direct consequence of Lemma 3.3 from [5] applied on the Hilbert space TM_{p_ε} to linear forms $D(F)_{p_\varepsilon}$ and $D(G_i)_{p_\varepsilon}$ $(1 \leqslant i \leqslant i(p_\varepsilon))$. This completes the proof.

Proof of Theorem 2. Using the density of C^∞-paths among L_1^2-paths (see, e.g., [5], Lemma 6.1), it is sufficient to obtain the conclusions of

Theorem 2 with $c_\varepsilon \in L_1^2([0, 1]; M)$. We can derive this fact by replacing in Theorem 3 the manifold M by the Sobolev manifold $L_1^2([0, 1]; M)$, and the function F by the energy integral. We claim that in this case the regularity condition (6) holds. This claim is easily verified by (2) and the fact that the mapping $c \to (c(0), c(1))$ from $L_1^2([0, 1]; M)$ to $M \times M$ is a submersion (see [8]).

3. An estimation of the distance. A basic problem related closely to the existence of minimal geodesics joining two closed subsets S_0 and S_1 of M which was discussed in Section 2 is to estimate the distance $d(S_0, S_1)$ from S_0 to S_1. To this aim we proceed by adapting a result of Ioffe [9] from Banach spaces to Riemannian manifolds. In our approach we essentially use the notions of generalized gradient and normal cone on M introduced in Definitions 1 and 2 of Section 2.

THEOREM 4. *Assume that M is a complete Riemannian manifold and f: $M \to R$ is a locally Lipschitz function on M. Let $S_0 = f^{-1}(0)$ and let S_1 be a nonempty closed subset of M such that $S_0 \cap S_1 = \emptyset$. Then for each $p \in S_1$ and $a > 0$ there exists $q \in S_1$ such that*

$$d(q, p) \leqslant a\, d(S_0, S_1)$$

and there exist $\omega \in \partial|f(q)|$ and $\theta \in N_{S_1}(q)$ satisfying

$$d(S_0, S_1) \leqslant |f(p)|/a\, \|\omega + \theta\|_q^*. \tag{12}$$

Proof. The argument is inspired from the proof of Theorem 16 of [6]. In fact, applying the variational principle of Ekeland to the function $|f|$ restricted to S_1 we get a point $q \in S_1$ with the properties $d(q, p) \leqslant a\, d(S_0, S_1)$ and

$$|f(x)| \geqslant |f(q)| - (|f(p)|/a\, d(S_0, S_1))\, d(x, q)$$

for all $x \in S_1$.

In particular, take $x = \exp_q(v)$ with $v \in \exp_q^{-1}(S_1)$ and

$$d(\exp_q(v), q) = \|v\|_q$$

(this condition holds if $\|v\|_q$ is sufficiently small). Then we obtain

$$\big|f(\exp_q(v))\big| + (|f(p)|/a\, d(S_0, S_1))\, d(\exp_q(v), q) \geqslant |f(q)|. \tag{13}$$

Thus the function of $v \in TM_q$ on the left-hand side of inequality (13) attains its minimum on $\exp_q^{-1}(S_1)$ at $O_q \in TM_q$. Now, in view of Definitions 1 and 2 the theorem follows by applying the Lagrange multiplier rule (see [2]) on the Hilbert space TM_q.

Remark 4. The previous result is not of local type since for a given point $p \in S_1$ it is not possible in general to find a chart at p whose domain intersects S_0.

Remark 5. If p is chosen to be an interior point of S_1, then for $a > 0$ small enough q is also an interior point of S_1, and hence $N_{S_1}(q) = \{0\}$. Then the estimation (12) takes the form

$$d(S_0, S_1) \leqslant |f(p)|/a \|\omega\|_q^*, \tag{14}$$

where $p \in S_1$ satisfies $d(q, p) \leqslant a d(S_0, S_1)$ and $\omega \in \partial |f(q)|$.

Remark 6. According to [4], every closed subset S_0 of M (in the case where M is separable) is the set of zeros of a smooth function on M. Hence (12) (or (14)) is in fact an estimation of the distance between two arbitrary closed subsets of a Riemannian manifold.

Theorem 4 yields, by an argument following the pattern of [9] (or [6], Corollary 17), a more precise estimation of $d(S_0, S_1)$.

COROLLARY 2. *Let f: $M \to R$ be a locally Lipschitz function and let S_1 be a nonempty closed subset of M. Let $S_0 = f^{-1}(0)$ and suppose $S_0 \cap S_1 = \emptyset$. Assume there exist points $x_i \in S_i$ $(i = 0, 1)$ and constants $\varepsilon > 0$ and $C > 0$ such that*

(i) *$d(x_0, x_1) \leqslant d(S_0, S_1) + \varepsilon$;*

(ii) *if $\omega \in \partial |f(x)|$ and $\theta \in N_{S_1}(x)$, then $\|\omega + \theta\|_x^* \geqslant C$ for all $x \in S_1$ with $d(x_0, x) \leqslant 2(d(S_0, S_1) + \varepsilon)$.*

Then $d(S_0, S_1) \leqslant |f(x_1)|/C$.

4. A fixed point theorem. We end this paper with a version of the Clarke's fixed point theorem [3] for the case of complete Riemannian manifolds. The novelty of our approach is to replace in the fixed point theorem of Clarke the directional contraction (see also [6]) by a contraction along a minimal geodesic.

THEOREM 5. *Let M be a complete Riemannian manifold with metric d and let S be a closed subset of M. Suppose f: $S \to S$ is a continuous mapping satisfying the following assumptions:*

(i) *for every $p \in S$ there exists a minimal geodesic c: $[0, 1] \to M$ beginning at p and ending at $f(p)$ and completely contained in S;*

(ii) *there exists $\sigma \in [0, 1)$ such that for every $p \in S$ there is $t_p \in (0, 1]$ satisfying*

$$d\big(f(c(t_p)), f(p)\big) \leqslant \sigma d\big(c(t_p), p\big).$$

Then there exists a point $q \in S$ such that $f(q) = q$.

Proof. Assume $f(p) \neq p$ for every $p \in S$. Choose $\varepsilon > 0$ such that $\sigma + \varepsilon < 1$. Define a continuous function F: $S \to R$ by $F(p) = d(f(p), p)$. By the variational principle of Ekeland, for $\varepsilon > 0$ there is some $q \in S$ such that

$$F(q) \leqslant F(p) + \varepsilon d(p, q) \quad \text{for all } p \in S.$$

Taking $p = c(t_q)$, where c is the minimal geodesic from (i) corresponding to the point q, we get

$$d(f(q), q) \leqslant d(f(c(t_q)), c(t_q)) + \varepsilon d(c(t_q), q).$$

Then, by (ii), we obtain

$$d(f(q), q) - d(f(q), c(t_q)) \leqslant (\sigma+\varepsilon) d(c(t_q), q). \tag{15}$$

Since $c(t)$ is a minimal geodesic, we have

$$d(f(q), q) = \int_0^1 \|c'(t)\|_{c(t)}\, dt.$$

Therefore

$$d(f(q), q) - d(f(q), c(t_q)) = d(q, c(t_q)),$$

and so from (15) we obtain

$$(1-\sigma-\varepsilon) d(q, c(t_q)) \leqslant 0.$$

Hence $c(t_q) = q$, which contradicts the minimality of the geodesic c. This proves the theorem.

Remark 7. The referee observed that a more general version of Theorem 5 is true in any complete metric space.

THEOREM 5′. *Let S be a complete metric space and let $f\colon S \to S$ be a continuous mapping satisfying the following condition: there is $\sigma \in [0, 1)$ such that if $p \in S$ with $p \neq f(p)$, then there exists $q \in S$, $q \neq p$, such that*

$$d(p, q) + d(q, f(p)) = d(p, f(p)) \quad \text{and} \quad d(f(p), f(q)) \leqslant \sigma d(p, q).$$

Then the mapping f has a fixed point.

The proof of Theorem 5′ goes the same lines as the proof of Theorem 5 and therefore we omit it.

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