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LINEAR MODULUS OF A MULTIDIMENSIONAL SEMIGROUP OF L_1-CONTRACTIONS AND A DIFFERENTIATION THEOREM

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1. Introduction. The purpose of this paper is two-fold: first we obtain a linear modulus semigroup of a multidimensional strongly continuous semigroup of L_1-contractions, and second, by applying this linear modulus semigroup we prove the differentiation theorem for multidimensional additive processes. Linear modulus of a linear contraction T on L_1-spaces was obtained first by Chacon and Krengel [4]. The extension of their result to semigroup of L_1-contractions is due to Kubokawa [9] and Sato [11]. Here we generalize this notion to encompass a multidimensional strongly continuous semigroup of L_1-contractions via a technique known as "reduction of dimension" (see [2], [5], [6] and [12]). The first operator theoretical local ergodic theorem (called also differentiation theorem) was obtained independently by Krengel [8] and Ornstein [10], and its multidimensional version was given by Terrell [12]. They considered the additive processes of the form

$$F_t = \int_0^t T_s f\,ds \quad \text{or} \quad F_{(u,v)} = \int_0^u \int_0^v U_{(t,r)} f\,dr\,dt,$$

where $f \in L_1$, and $\{T_s\}_{s\geq 0}$ and $\{U_{(t,r)}\}_{t,r\geq 0}$ are semigroups of L_1-contractions. Akcoglu and Krengel [3] proved the local ergodic theorem for arbitrary additive processes with respect to a strongly continuous semigroup $\{T_t\}_{t>0}$ of positive L_1-contractions, and Akcoglu and delJunco [2] generalized this result to the multidimensional case. Recently, Akcoglu and Falkowitz [1] obtained the differentiation theorem of Akcoglu–Krengel [3] without the positivity hypothesis of $\{T_t\}_{t>0}$. Our result generalizes the theorem of Akcoglu–delJunco as well as the theorem of Akcoglu–Falkowitz. This result has also been obtained by Emilion [7]. The method in [7], however, employs a process of reduction of dimension which does not yield a one-dimensional semigroup but a subsemigroup. Then, by introducing "almost additive processes", the result is being obtained via a technique due to Akcoglu and Falkowitz [1]. On the other hand, with the help of the linear modulus semigroup we obtained, we dominate the multidimensional additive process by a one-dimensional additive process. Then using the properties of the semigroups and the additive processes we obtain a direct proof.

Let (X, μ) be a σ-finite measure space and $L_1 = L_1(X, \mu)$ be the classical Banach space of real-valued integrable functions on X. Let L_1^+ denote the positive cone of L_1, and we will use the notation

$$f \vee g = \max\{f, g\} \quad \text{and} \quad f \wedge g = \min\{f, g\}, \quad f, g \in L_1.$$

Let $\boldsymbol{R}^2$ be the usual two-dimensional real vector space, considered with all its usual structure. The positive cone of it is $\boldsymbol{R}_+^2$ and $C = \operatorname{int} \boldsymbol{R}_+^2$. In particular, $\boldsymbol{R}_+^2$ is ordered in the usual way, that is, for any $(u, v), (t, r) \in \boldsymbol{R}_+^2$,

$$(u, v) \leqslant (t, r) \quad \text{if } u \leqslant t \text{ and } v \leqslant r,$$

$$(u, v) < (t, r) \quad \text{if } (u, v) \leqslant (t, r) \text{ and } (u, v) \neq (t, r).$$

For any $s \in \boldsymbol{R}$, $\mathbf{s}$ will denote the vector $\mathbf{s} = (s, s) \in \boldsymbol{R}^2$.

Consider a two-dimensional strongly continuous semigroup of L_1-contractions $U = \{U_{(t,r)}\}_{(t,r) \in \boldsymbol{R}_+^2}$, that is:

(1.1) For each $(t, r) \in \boldsymbol{R}_+^2$, $U_{(t,r)}$ is a linear operator on L_1 with

$$\|U_{(t,r)}\|_1 \leqslant 1.$$

(1.2) For each $(t, r), (u, v) \in \boldsymbol{R}_+^2$,

$$U_{(t,r)} U_{(u,v)} = U_{(t+u, r+v)}.$$

(1.3) For all $f \in L_1$ and $(t, r), (u, v) \in C$,

$$\lim_{(t,r) \to (u,v)} \|U_{(u,v)} f - U_{(t,r)} f\|_1 = 0.$$

U is called *continuous at the origin* if, in addition,

(1.4) for all $f \in L_1$ and $(t, r) \in \boldsymbol{R}_+^2$,

$$\lim_{(t,r) \to 0^+} \|U_{(t,r)} f - U_0 f\|_1 = 0.$$

U is called a *Dunford–Schwartz semigroup* if it satisfies

(1.5) For each $(t, r) \in \boldsymbol{R}_+^2$, $\|U_{(t,r)}\|_\infty \leqslant 1$ in addition to (1.1)–(1.3).

A family of functions $F = \{F_{(u,v)}\}_{(u,v) \in C}$ is called a (two-dimensional) U-*additive process* (see [2], [3], [5]) if, for each $(u, v), (t, r) \in C$ with $0 < (t, r) \leqslant (u, v)$,

$$F_{(u,v)} = \begin{cases} F_{(t,v)} + U_{(t,0)} F_{(u-t,v)} & \text{if } t < u, \\ F_{(u,r)} + U_{(0,r)} F_{(u,v-r)} & \text{if } r < v. \end{cases} \tag{1.6}$$

Such a process is called *bounded* if

(1.7) $$\sup_{(u,v)>0} \frac{1}{uv} \int F_{(u,v)}\, d\mu = \gamma_F < \infty .$$

The constant γ_F is called the *time constant* of the process.

In what follows, we will consider U as being a strongly continuous semigroup of L_1-contractions which is also continuous at the origin, and F as being a bounded U-additive process. Here it should be indicated that each $F_{(u,v)}$ is an element of L_1, not an actual function. Since our aim is to prove the differentiation theorem for F, in other words, to show that $\lim_{u\to 0^+} u^{-2} F_u$ exists a.e., we either should choose suitable representatives $F_{(u,v)}(x)$ or should let u range over a countable dense subset of $\boldsymbol{R}$ (see [2], [3], [5]). For convenience, we will let u approach 0 through positive rationals and denote this by a $q\text{-}\lim_{u\to 0^+}$, which is no loss of generality (see [2] and [3]). Also, all the proofs will be given in the two-dimensional case for brevity, since the extension of the results to the arbitrary n-dimensional case ($n \geqslant 2$) is straightforward.

2. Linear modulus semigroup. In this section we will first obtain a method which will enable us to reduce the two-dimensional case to the one-dimensional one, and then we will construct a linear modulus of the reduced semigroup of operators.

(2.1) *Reduction of dimension.* This method is originally due to Dunford and Schwartz [6] in an implicit form and later developed by Terrell [12], Akcoglu and delJunco [2] and Çomez [5] for positive operators. Here, we will make use of their idea to obtain a similar result for an arbitrary (not necessarily positive) semigroup of operators.

Let, for any $x \in (0, \infty)$ and $\beta \in \boldsymbol{R}$,

$$\Phi_x(\beta) = \begin{cases} \dfrac{x}{2\sqrt{\pi}} \beta^{-3/2} \exp\left\{-\dfrac{x^2}{4\beta}\right\} & \text{if } \beta > 0, \\ 0 & \text{if } \beta \leqslant 0. \end{cases}$$

For $(u, v) \in \boldsymbol{R}^2$, define $\Phi_x(u, v) = \Phi_x(u)\Phi_x(v)$. Then, for each fixed $x \in (0, \infty)$, $\Phi_x\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ is a nonnegative continuous function that vanishes on $\boldsymbol{R}^2 \setminus \boldsymbol{R}^2_+$ (see [2], [6], [12]). Also

$$\int_{\boldsymbol{R}^2} \Phi_x(u, v)\, du\, dv = 1 \quad \text{and} \quad \int_{\boldsymbol{R}^2} \Phi_x(t-u, r-v)\, \Phi_y(u, v)\, du\, dv = \Phi_{x+y}(t, r)$$

for each $x, y \in (0, \infty)$ and $(t, r) \in \boldsymbol{R}^2$ (see [6] and [12]). For any $x \in (0, \infty)$ and $f \in L_1$, define

(2.2) $$V_x f = \int_{\boldsymbol{R}^2} \Phi_x(u, v)\, U_{(u,v)} f\, du\, dv .$$

Then we have the following properties of V_x (see [6], [12]):

(2.3) V_x is linear on L_1 for each $x > 0$.

(2.4) $V_x V_y = V_{x+y}$ for each $x, y \in (0, \infty)$.

(2.5) $\|V_x\|_1 \leq 1$ (and $\|V_x\|_\infty \leq 1$ if $\|U_{(u,v)}\|_\infty \leq 1$) for each $x \in (0, \infty)$.

That is, the family $V = \{V_x\}_{x>0}$ is a semigroup of L_1-contractions on L_1. Moreover, this semigroup is strongly continuous and continuous also at the origin with $V_0 = U_0$ [12] (and is a Dunford–Schwartz semigroup if U is so).

Consider the bounded U-additive process $F = \{F_{(u,v)}\}_{(u,v)} \in C$. For any $I \subset \boldsymbol{R}^2$ with $I = [a_1, b_1] \times [a_2, b_2]$, $a_i, b_i \in \boldsymbol{R}$, $i = 1, 2$, define (see [2] and [5])

$$\tilde{F}_I = U_{(a_1, a_2)} F_{(b_1 - a_1, b_2 - a_2)}.$$

Let ϕ: $\boldsymbol{R}^2 \to \boldsymbol{R}$ be a continuous function and let $J \subset \boldsymbol{R}^2_+$ be a bounded interval, where $J = J_1 \times J_2$. Then

$$\int_J \phi(u, v) \tilde{F}(du, dv) = \int_{J_1} \int_{J_2} \phi(u, v) \tilde{F}(du, dv)$$

is defined as the L_1-limit of the Riemann sums

$$\sum \phi(u^i, v^i) F(J^i_1 \times J^i_2),$$

where $\{J^i\}$ are finite partitions of J into intervals with $(u^i, v^i) \in J^i$. In this case, if we have

$$\int_{\boldsymbol{R}^2_+} |\phi(u, v)| \, du dv < \infty,$$

then $\int_{\boldsymbol{R}^2_+} \phi(u, v) \tilde{F}(du, dv)$ is well defined since, for any $J \subset \boldsymbol{R}^2_+$,

$$\left\| \int_J \phi(u, v) \tilde{F}(du, dv) \right\|_1 \leq \gamma_F \int_J |\phi(u, v)| \, du dv$$

(see [2]). Also observe that, for any $(t, r) \in \boldsymbol{R}^2_+$,

$$U_{(t,r)} \int_{\boldsymbol{R}^2_+} \phi(u, v) \tilde{F}(du, dv) = \int_{\boldsymbol{R}^2_+} \phi(u-t, v-r) \tilde{F}(du, dv).$$

Notice that the function $\Phi_x(u, v)$ defined at the beginning of this section satisfies all the properties. Hence, for any $x > 0$, put

$$h_x = \int_{\boldsymbol{R}^2_+} \Phi_x(u, v) \tilde{F}(du, dv)$$

and, consequently, define a new process $H = \{H_a\}_{a>0}$ by

$$H_a = \int_0^a h_x dx, \quad a > 0. \tag{2.6}$$

Then, for any $y > 0$,

$$V_y H_a = V_y \int_0^a h_x dx = \int_0^a V_y h_x dx = \int_0^a [V_y \int_{\mathbf{R}^2} \Phi_x(u, v) \tilde{F}(du, dv)] dx$$

$$= \int_0^a \{ \int_{\mathbf{R}^2} \Phi_y(t, r) U_{(t,r)} [\int_{\mathbf{R}^2} \Phi_x(u, v) \tilde{F}(du, dv)] dtdr \} dx$$

$$= \int_0^a \{ \int_{\mathbf{R}^2} [\int_{\mathbf{R}^2} \Phi_y(t, r) \Phi_x(u-t, v-r) dtdr] \tilde{F}(du, dv) \} dx$$

$$= \int_0^a [\int_{\mathbf{R}^2} \Phi_{x+y}(u, v) \tilde{F}(du, dv)] dx = \int_0^a h_{x+y} dx = \int_y^{y+a} h_x dx.$$

Therefore

$$H_{a+y} = \int_0^{a+y} h_x dx = \int_0^y h_x dx + \int_y^{y+a} h_x dx = H_y + V_y H_a$$

for any $y > 0$, $a > 0$. Therefore, $H = \{H_a\}_{a>0}$ is a V-additive process. Moreover, from the construction it is immediate that H is a bounded process with $\gamma_H \leqslant \gamma_F$.

(2.7) *Linear modulus.* Starting from the strongly continuous semigroup V obtained in the previous section, we will obtain a positive semigroup of operators which is its linear modulus using a technique very similar to that of Sato [11] and Kubokawa [9]. Indeed, we have the following

(2.8) THEOREM. *Let $V = \{V_x\}_{x \geqslant 0}$ be the strongly continuous semigroup in* (2.1). *Then there exists a strongly continuous semigroup $T = \{\tau_x\}_{x \geqslant 0}$ of positive L_1-contractions, called the linear modulus of V, such that*

(2.9) $$|V_x f| \leqslant \tau_x |f| \quad \textit{for all } f \in L_1 \textit{ and for all } x > 0.$$

Proof. Fix $x > 0$ and consider the interval $[0, x]$. Let

$$P = \{x_i \in [0, x]: i = 0, 1, \ldots, n,\ 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = x\}$$

be a partition of $[0, x]$. For any $r > 0$ and $f \in L_1^+$, define

$$|V_r| f = \int_{\mathbf{R}^2} \Phi_r(u, v) |U_{(u,v)}| f du dv,$$

where $|U_{(u,v)}|$ is the linear modulus of $U_{(u,v)}$ for each $(u, v) \in \mathbf{R}_+^2$ in the sense of Chacon–Krengel [4]. Then, for any $f \in L_1^+$, let

$$D(P, f) = |V_{x_n - x_{n-1}}| \ldots |V_{x_2 - x_1}| |V_{x_1}| f = \prod_{i=0}^{n-1} |V_{x_{i+1} - x_i}| f.$$

Let P' be another partition of $[0, x]$ which is finer than P, say by one point

$a > 0$. Assume $x_i < a < x_{i+1}$ for some i $(0 \leqslant i \leqslant n)$. Then, for $f \in L_1^+$,

$$D(P', f) = |V_{x_n - x_{n-1}}| \ldots |V_{x_{i+1} - a}| |V_{a - x_i}| \ldots |V_{x_1}| f.$$

Now,

$$|V_{x_{i+1} - x_i}| f = |V_{x_{i+1} - a + a - x_i}| f = \int_{\mathbf{R}^2} \Phi_{(x_{i+1} - a) + (a - x_i)}(u, v) |U_{(u,v)}| f du dv$$
$$= \int_{\mathbf{R}^2} \Big[\int_{\mathbf{R}^2} \Phi_{a - x_i}(u - \alpha, v - \beta) \Phi_{x_{i+1} - a}(\alpha, \beta) |U_{(u,v)}| f du dv \Big] d\alpha d\beta$$
$$= \int_{\mathbf{R}^2} \Big[\int_{\mathbf{R}^2} \Phi_{a - x_i}(u - \alpha, v - \beta) \Phi_{x_{i+1} - a}(\alpha, \beta) |U_{(u,v)}| f du dv \Big] d\alpha d\beta$$
$$= \int_{\mathbf{R}^2} \Big[\int_{\mathbf{R}^2} \Phi_{a - x_i}(u, v) \Phi_{x_{i+1} - a}(\alpha, \beta) |U_{(u+\alpha, v+\beta)}| du dv \Big] d\alpha d\beta.$$

Since $|U_{(u+\alpha, v+\beta)}| \leqslant |U_{(\alpha,\beta)}| |U_{(u,v)}|$ (see [4]), we have

$$|V_{x_{i+1} - x_i}| f \leqslant \int_{\mathbf{R}^2} \Big[\int_{\mathbf{R}^2} \Phi_{a - x_i}(u, v) \Phi_{x_{i+1} - a}(\alpha, \beta) |U_{(\alpha,\beta)}| |U_{(u,v)}| f du dv \Big] d\alpha d\beta$$
$$= \int_{\mathbf{R}^2} \Phi_{x_{i+1} - a}(\alpha, \beta) |U_{(\alpha,\beta)}| \Big[\int_{\mathbf{R}^2} \Phi_{a - x_i}(u, v) |U_{(u,v)}| \Phi du dv \Big] d\alpha d\beta$$
$$= |V_{x_{i+1} - a}| \Big(\int_{\mathbf{R}^2} \Phi_{a - x_i}(u, v) |U_{(u,v)}| f du dv \Big) = |V_{x_{i+1} - a}| (|V_{a - x_i}| f).$$

Hence we see that $D(P', f) \geqslant D(P, f)$. For any $x > 0$, let P_x denote the family of all finite partitions P of $[0, x]$ as above. If $P, P' \in P_x$, we will write $P < P'$ if P' is a refinement of P. With this partial order, P_x is an increasingly filtered set. Since each $|U_{(u,v)}|$ is an L_1-contraction, we see also that $\|D(P, f)\|_1 \leqslant \|f\|_1$ for each $P \in P_x$ and $f \in L_1^+$. Therefore, for $f \in L_1^+$, define

$$\text{(2.10)} \qquad \tau_x f = \sup_{P \in P_x} D(P, f) = \lim_{P \in P_x} D(P, f), \quad x > 0,$$
$$\tau_0 = |V_0|.$$

Then τ_x is well defined for each $x \geqslant 0$ and is a linear contraction on L_1^+, and hence on L_1.

Given $x > 0$, $y > 0$, let

$$P_1 = \{0 = x_0 < x_1 < \ldots < x_n = x\} \quad \text{and} \quad P_2 = \{0 = y_0 < y_1 < \ldots < y_m = y\}$$

be partitions of $[0, x]$ and $[0, y]$, respectively. Then

$$P = \{0 = x_0 < x_1 < \ldots < x_n = x_n + y_0 < x_n + y_1 < \ldots < x_n + y_m = x + y\}$$

is a partition of $[0, x+y]$. Therefore, for $f \in L_1^+$,

$$D(P, f) = \prod_{i=0}^{m-1} |V_{y_{i+1}-y_i}| \Big(\prod_{j=0}^{n-1} |V_{x_{j+1}-x_j}|\Big) f$$

and $\tau_{x+y} f = \tau_y \tau_x f$ by taking the limit over partitions $P \in P_{x+y}$ which contains x.

To prove the strong continuity, we will first show that $\{|V_x|\}_{x \geqslant 0}$ is strongly continuous. Let $0 < x < y$. Then, for any $f \in L_1$,

$$\begin{aligned} \| |V_x| f - |V_y| f \|_1 &= \int [| \int_{\mathbf{R}^2} (\Phi_x(u, v) - \Phi_y(u, v)) |U_{(u,v)}| f du dv |] \, d\mu \\ &\leqslant \|f\|_1 \int_{\mathbf{R}^2} |\Phi_x(u, v) - \Phi_y(u, v)| \, du dv, \end{aligned}$$

since $|U_{(u,v)}|$ is an L_1-contraction for each $(u, v) \in \mathbf{R}_+^2$. Thus

$$\lim_{y \to x} \| |V_y| f - |V_x| f \|_1 = 0$$

since (see [12])

$$\lim_{y \to x} \int_{\mathbf{R}^2} |\Phi_x(u, v) - \Phi_y(u, v)| \, du dv = 0.$$

Then, applying exactly the same method as in [11], we obtain

$$\lim_{y \to x} \|\tau_y f - \tau_x f\| = 0.$$

Here, we can also choose $x = 0$ (see [9] and [11]). Hence $T = \{\tau_x\}_{x \geqslant 0}$ is a positive strongly continuous semigroup of L_1-contractions which is also continuous at the origin.

It remains to prove (2.9). For, let $f \in L_1$ and $x > 0$. Then

$$\begin{aligned} |V_x f| &= \Big| \int_{\mathbf{R}^2} \Phi_x(u, v) \, U_{(u,v)} f du dv \Big| \\ &\leqslant \int_{\mathbf{R}^2} \Phi_x(u, v) |U_{(u,v)} f| \, du dv \\ &\leqslant \int_{\mathbf{R}^2} \Phi_x(u, v) |U_{(u,v)}| \, |f| \, du dv \quad (\text{since } |U_{(u,v)} f| \leqslant |U_{(u,v)}| \, |f| \text{ by [4]}) \\ &= |V_x| \, |f| \leqslant \tau_x |f| \quad (\text{by (2.10)}). \end{aligned}$$

Note that if U is a Dunford–Schwartz semigroup (i.e., $\|U_{(t,r)}\|_\infty \leqslant 1$ for each $(t, r) \in \mathbf{R}_+^2$), then so is T.

The following result is a modified version of a lemma of Terrell [12] which is originally due to Dunford and Schwartz [6].

(2.11) LEMMA. *Let U, V and T be as above and $f \in L_1$. Then there exists a* *constant* $\delta > 0$, *independent of* f *and* U, *such that, for every* $u > 0$,

$$\frac{\delta}{u^2}\left|\int_0^u\int_0^u U_{(t,r)} f dt dr\right| \leqslant \frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} \tau_x |f| dx. \tag{2.12}$$

Proof. It is shown in [12] that there exists a constant $\delta > 0$ such that if $u > 0$, then

$$\frac{\delta}{u^2} < \frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} \Phi_x(t, r) dx$$

whenever $0 < t$, $r < u$. Now,

$$\begin{aligned}
\frac{\delta}{u^2}\left|\int_0^u\int_0^u U_{(t,r)} f dt dr\right| &\leqslant \int_0^u\int_0^u \frac{\delta}{u^2} |U_{(t,r)}| |f| dt dr \\
&\leqslant \int_0^u\int_0^u \left[\frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} \Phi_x(t, r) |U_{(t,r)}| |f| dx\right] dt dr \\
&= \int_0^u\int_0^u \left[\frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} \Phi_x(t, r) |U_{(t,r)}| |f| dx\right] dt dr \\
&= \frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} \left[\int_0^u\int_0^u \Phi_x(t, r) |U_{(t,r)}| |f| dt dr\right] dx \\
&\leqslant \frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} \left[\int_{R^2} \Phi_x(t, r) |U_{(t,r)}| |f| dt dr\right] dx \\
&= \frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} |V_x| |f| dx \leqslant \frac{1}{\sqrt{u}} \int_0^{\sqrt{u}} \tau_x |f| dx
\end{aligned}$$

by (2.9).

(2.13) Remark. The method of reduction of dimension in (2.1) is defined in general for the dimension $n \geqslant 2$ when it is an even integer. But this is not a loss of generality as noted in [2], p. 758.

(2.14) *Basic properties of F and H.* First, we recall the following result due to Akcoglu and Falkowitz [1] (Theorem 2.1):

(2.15) Theorem. *Let* $\{F_t\}_{t>0}$ *be a bounded* $\{T_t\}_{t\geqslant 0}$*-additive process, where* $\{T_t\}_{t\geqslant 0}$ *is a strongly continuous semigroup of* L_1*-contractions. Let* $\{\tau_t\}_{t\geqslant 0}$ *be a linear modulus of* $\{T_t\}$. *Then there is a* $\{\tau_t\}_{t\geqslant 0}$*-additive process* $\{G_t\}_{t>0}$ *such that*

(i) $|F_t| \leqslant G_t$ *a.e. for every* $t > 0$,

(ii) $\{G_t\}_{t>0}$ *has the same bound as* $\{F_t\}$.

(2.16) Remark. It should be noted that the linear modulus $\{\tau_t\}_{t>0}$ in Theorem (2.15) is in the sense of Kubokawa–Sato. However, the linear modulus we have defined in Section (2.7) has the same properties as that of Kubokawa–Sato. Hence Theorem (2.15) yields that there exists a T-additive process (T is as in Theorem (2.8)) $\{G_u\}_{u>0}$ such that

(i) $|H_u| \leqslant G_u$ a.e. for each $u > 0$,

(ii) $\gamma_H = \gamma_G$.

Before stating the next result we introduce the following notation for convenience: for any $(t, r) \in C$ or $u > 0$, let

$$f_{(t,r)} = \frac{1}{tr} F_{(t,r)} \quad \text{and} \quad g_u = \frac{1}{u} G_u. \tag{2.17}$$

By s-lim or w-lim we will mean limit taken in the strong operator or weak topology of L_1, respectively.

(2.18) LEMMA. *For any* $(u, v) \in C$,

$$F_{(u,v)} = \operatorname*{s\text{-}lim}_{(\alpha,\beta)\to 0^+} \int_0^u \int_0^v U_{(t,r)} f_{(\alpha,\beta)}\, dr dt.$$

This result has been obtained by Akcoglu and Falkowitz [1] in the 1-dimensional case. For F, which is an additive process with respect to a positive semigroup of operators U, it is obtained by Akcoglu and delJunco [2], Lemma (3.2). However, in their proof, positivity of the operators does not play any role, hence we obtain the lemma by applying their method of proof to our case. That is why we do not provide the proof.

3. The differentiation theorem. We start by stating a technical lemma (without proof) which is due to Akcoglu and Falkowitz [1] (Lemma 2.1):

(3.1) LEMMA. *Let* $\phi_n \in L_1$ *and* $|\phi_n| \leqslant \psi_n \in L_1^+$ *be such that there exists* $\psi \in L_1^+$ *with*

$$\|\psi_n - \psi\|_1 \to 0 \quad \text{as } n \to \infty.$$

Then $\{\phi_n\}$ *is weakly-sequentially compact.*

Let $T = \{\tau_x\}_{x\geqslant 0}$ and $\{G_u\}_{u>0}$ be as in Remark (2.16). Then, by the theorem of Akcoglu–Krengel [3],

(3.2) $q\text{-}\lim_{u\to 0^+} u^{-1} G_u = g$ exists a.e., $g \in L_1$.

Define $G_u = G'_u + G''_u$ as in [1] and [3], where

$$G'_u = \int_0^u \tau_x g\, dx$$

and $\{G''_u\}_{u>0}$ is a positive T-additive process with

$$q\text{-}\lim_{u\to 0^+} u^{-1} G''_u = 0 \text{ a.e.}$$

Now, we state and prove the main result:

(3.3) THEOREM. *Let $U = \{U_{(t,r)}\}_{(t,r)\in \mathbf{R}^2_+}$ be a strongly continuous semigroup of L_1-contractions which is also continuous at the origin. If $F = \{F_{(u,v)}\}_{(u,v)\in C}$ is a bounded U-additive process, then $q\text{-}\lim_{u\to 0^+} u^{-2} F_u$ exists a.e.*

Proof. Let $\varepsilon_n \to 0^+$ be any positive sequence of reals and consider

$$\psi_n = g_{\varepsilon_n} \wedge g,$$

where g is as defined in (3.2). Therefore, $0 \leqslant \psi_n \leqslant g$ and, obviously, $\psi_n \to g$ a.e. [3]. Now, let

$$\phi_n = (-\delta\psi_n) \vee (f_{\varepsilon_n} \wedge \delta\psi_n),$$

where $\delta > 0$ is the constant in Lemma (2.11). Thus $|\phi_n| \leqslant \delta\psi_n$. Notice that, by the Bounded Convergence Theorem, $\|\psi_n - g\|_1 \to 0$ as $n \to \infty$. Hence all the conditions in Lemma (3.1) are satisfied. Thus, by passing to a subsequence if necessary, we may assume that

$$\text{w-}\lim \phi_n = \tilde{f} \in L_1.$$

Put $f = U_0 \tilde{f}$ and define, for each $(u, v) \in C$,

$$F'_{(u,v)} = \int_0^u \int_0^v U_{(t,r)} f dr dt,$$

and let

$$F''_{(u,v)} = F_{(u,v)} - F'_{(u,v)}.$$

Obviously, $\{F'_{(u,v)}\}$ and $\{F''_{(u,v)}\}$ are U-additive processes.

Let $S_\alpha f = \alpha^{-2} F'_\alpha$. Then (see [8])

$$\lim_{u\to 0^+} u^{-2} \int_0^u \int_0^u U_{(t,r)} S_\alpha f dt dr = U_0 S_\alpha f \text{ a.e.} \tag{3.4}$$

(i.e., $S_\alpha f$ is regular in the sense of [8]). Therefore, if

$$\bar{f} = \overline{\lim_{u\to 0^+}} |S_u f - U_0 f|,$$

then

$$\bar{f} \leqslant \overline{\lim_{u\to 0^+}} |S_u(U_0 f - S_\alpha f)| + \overline{\lim_{u\to 0^+}} |S_u(S_\alpha f) - U_0(S_\alpha f)| + |S_\alpha f - U_0 f|$$

$$= \overline{\lim_{u\to 0^+}} |S_u(U_0 f - S_\alpha f)| + |S_\alpha f - U_0 f| \quad \text{(by (3.4))}$$

$$\leqslant \overline{\lim_{u\to 0^+}} \frac{1}{\delta\sqrt{u}} \int_0^{\sqrt{u}} \tau_x |U_0 f - S_\alpha f| dx + |S_\alpha - U_0 f| \quad \text{(by Lemma (2.11))}$$

$$= \delta^{-1}\tau_0 |U_0 f - S_\alpha f| + |S_\alpha f - U_0 f|$$

(by the Krengel–Ornstein theorem [8], [10]).

Consequently, $\|\bar{f}\| \leqslant (1/\delta + 1)\|S_\alpha f - U_0\|_1 \to 0$ by strong continuity. Hence $\bar{f} \equiv 0$ a.e., which gives

$$\lim_{u\to 0^+} u^{-2} F'_u = U_0 \tilde{f} = f \text{ a.e.}$$

Hence, if we can show that

$$q\text{-}\lim_{u\to 0^+} u^{-2} F''_u = 0 \text{ a.e.},$$

the theorem will be proved.

First of all observe that, for each $n \geqslant 0$,

$$|f_{\varepsilon_n} - \phi_n| \leqslant \delta(g_{\varepsilon_n} - \psi_n).$$

Now, by Lemma (2.18),

$$F_{(u,v)} = \text{s-}\lim_{n\to\infty} \int_0^u \int_0^v U_{(t,r)} f_{\varepsilon_n} drdt, \qquad G_{\sqrt{u}} = \text{s-}\lim_{n\to\infty} \int_0^{\sqrt{u}} \tau_x g_{\varepsilon_n} dx.$$

Since both $\int_0^u \int_0^v U_{(t,r)}(\cdot) drdt$ and $\int_0^{\sqrt{u}} \tau_t(\cdot) dt$ are bounded linear operators, we obtain

$$\int_0^u \int_0^v U_{(t,r)} f drdt = \int_0^u \int_0^v U_{(t,r)} U_0 \tilde{f} drdt = \text{w-}\lim_{n\to\infty} \int_0^u \int_0^v U_{(t,r)} \phi_n drdt$$

and

$$\int_0^{\sqrt{u}} \tau_x g dx = \text{s-}\lim_{n\to\infty} \int_0^{\sqrt{u}} \tau_x \psi_n dx = \text{w-}\lim_{n\to\infty} \int_0^{\sqrt{u}} \tau_x \psi_n dx.$$

Consequently,

$$F''_{(u,v)} = F_{(u,v)} - \int_0^u \int_0^v U_{(t,r)} f drdt$$

$$= \left[\text{w-}\lim_{n\to\infty} \int_0^u \int_0^v U_{(t,r)} f_{\varepsilon_n} drdt\right] - \left[\text{w-}\lim_{n\to\infty} \int_0^u \int_0^v U_{(t,r)} \phi_n drdt\right]$$

$$= \text{w-}\lim_{n\to\infty} \int_0^u \int_0^v U_{(t,r)} (f_{\varepsilon_n} - \phi_n) drdt.$$

Therefore

$$\frac{1}{u^2}|F_u| = \frac{1}{u^2}\left|\text{w-}\lim_{n\to\infty}\int_0^u\int_0^u U_{(t,r)}(f_{\varepsilon_n}-\phi_n)\,dr\,dt\right|$$

$$\leqslant \text{w-}\lim_{n\to\infty}\frac{1}{u^2}\left|\int_0^u\int_0^u U_{(t,r)}(f_{\varepsilon_n}-\phi_n)\,dr\,dt\right|$$

$$\leqslant \text{w-}\lim_{n\to\infty}\frac{1}{\delta\sqrt{u}}\int_0^{\sqrt{u}}\tau_x|f_{\varepsilon_n}-\phi_n|\,dx$$

by Lemma (2.11). Consequently,

$$\frac{1}{u^2}|F''_u| \leqslant \text{w-}\lim_{n\to\infty}\frac{1}{\delta\sqrt{u}}\int_0^{\sqrt{u}}\delta\tau_x(g_{\varepsilon_n}-\psi_n)\,dx$$

$$= \text{w-}\lim_{n\to\infty}\frac{1}{\sqrt{u}}\int_0^{\sqrt{u}}\tau_x(g_{\varepsilon_n}-\psi_n)\,dx = \frac{1}{\sqrt{u}}(G_{\sqrt{u}}-G'_{\sqrt{u}}) = \frac{1}{\sqrt{u}}G''_{\sqrt{u}}.$$

Since $u^{-1}G''_u \to 0$ a.e. (cf. [3]) as $u \to 0^+$, we see that

$$u^{-2}|F''_u| \to 0 \text{ a.e.} \quad \text{as } u \to 0^+,$$

which proves the theorem.

(3.5) Remark. The results we have obtained are valid if U is a strongly continuous semigroup of L_1-contractions, not necessarily continuous at the origin as is shown in [1] for the one-dimensional case and in [2], [5], [7] for the multidimensional case.

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