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A POLYNOMIAL CHARACTERIZATION OF NONDISTRIBUTIVE MODULAR LATTICES

BY

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0. Introduction. An algebra $(A, +, \cdot)$ of type $(2, 2)$ is said to be a *bisemilattice* (see [16]) if it satisfies the following identities:

(i) $x+x = x$, $x \cdot x = x$;

(ii) $x+y = y+x$, $x \cdot y = y \cdot x$;

(iii) $(x+y)+z = x+(y+z)$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(in the sequel we shall write xy instead of $x \cdot y$).

The class of all algebras $(A, +, \cdot)$ of type $(2, 2)$ satisfying (i) and (ii) is denoted by $V(+, \cdot)$ and the class of all bisemilattices is denoted by $B(+, \cdot)$. By $p_n = p_n(\mathfrak{A})$ we denote the number of all essentially n-ary polynomials over $\mathfrak{A}$.

In his survey of equational logic, Taylor ([19], p. 41) poses a general problem asking whether the numbers $p_n(\mathfrak{A})$ characterize (to some extent and perhaps in special circumstances) the algebra $\mathfrak{A}$. Our main result can be treated as a contribution to this problem:

THEOREM. *Let $(A, +, \cdot)$ be a bisemilattice. Then $(A, +, \cdot)$ is a nondistributive modular lattice if and only if $(A, +, \cdot)$ has precisely* 19 *essentially ternary polynomials.*

Recall that a lattice $(L, +, \cdot)$ is *modular* if $x(xy+z) = xy+xz$ holds in $(L, +, \cdot)$ for all $x, y, z \in L$ (see [12]).

An algebra $\mathfrak{A} = (A, F)$ of type τ is called *proper* (see [3]) if all fundamental polynomials of $\mathfrak{A}$ are different and every nonnullary f from F depends on all its variables.

Let $f = f(x_1, \ldots, x_n)$ be a function on a set A. We say that *f admits a permutation* $\sigma \in S_n$ (where S_n denotes the symmetric group of n letters) of its variables if $f = f^\sigma$, where

$$f^\sigma(x_1, \ldots, x_n) = f(x_{\sigma 1}, \ldots, x_{\sigma n})$$

for all $x_1, \ldots, x_n \in A$. By $G(f)$ we denote the subgroup of S_n of all admissible permutations of f (see [14]). A function $f = f(x_1, \ldots, x_n)$ is *symmetric* if $f = f^\sigma$ for all $\sigma \in S_n$, and is *idempotent* if $f(x, \ldots, x) = x$ for all $x \in A$. Recall that an algebra (A, F) is *idempotent* (*symmetric*) if every $f \in F$ is idempotent (symmetric).

For undefined concepts used here we refer to [11].

Before proving the quoted theorem we present some recent results concerning the number of polynomials of algebras from the varieties $B(+, \cdot)$ and $V(+, \cdot)$.

1. $\{p_n\}$-sequences for algebras from $V(+, \cdot)$.

THEOREM 1.1 ([4]). *If $(A, +, \cdot)$ is a proper bisemilattice, then*

$$p_n(A, +, \cdot) \geqslant 2+n! \quad \textit{for all } n \geqslant 3.$$

THEOREM 1.2 ([3]). *Let $(A, +, \cdot)$ be a bisemilattice with* card $A \geqslant 2$. *Then $(A, +, \cdot)$ is a lattice if and only if*

$$p_2(A, +, \cdot) = 2.$$

THEOREM 1.3 ([3]). *There exists no bisemilattice $(A, +, \cdot)$ for which $p_2(A, +, \cdot) = 3$.*

Note that bisemilattices with four and five essentially binary polynomials are considered in [8] and [9].

THEOREM 1.4 ([7]). *Let $(A, +, \cdot)$ be a proper algebra from $V(+, \cdot)$. Then $(A, +, \cdot)$ is a distributive lattice if and only if*

$$p_3(A, +, \cdot) = 9.$$

Denote by $V_\infty(+, \cdot)$ the subvariety of the variety $V(+, \cdot)$ of all algebras $(A, +, \cdot)$ satisfying the identity

$$(x+y)y = x$$

(for details see [6]). Then we have

THEOREM 1.5 ([6]). *If $(A, +, \cdot) \in V_\infty(+, \cdot)$ and $(A, +, \cdot)$ is proper, then $(A, +, \cdot)$ contains infinitely many essentially n-ary polynomials for every $n \geqslant 2$.*

We should mention here that if $(A, +, \cdot) \in V_\infty(+, \cdot)$, card $A \geqslant 2$, and $(A, +, \cdot)$ is improper, then $(A, +, \cdot)$ is a Steiner quasigroup. Now, using Theorem 3 of [13] and the fact that such a Steiner quasigroup contains isomorphically as a subgroupoid the groupoid $(\{0, 1, 2\}, 2x \underset{3}{+} 2y)$, we infer that for any improper algebra $(A, +, \cdot) \in V_\infty(+, \cdot)$, where card $A \geqslant 2$, we have

$$p_n(A, +, \cdot) \geqslant \frac{2^n-(-1)^n}{3} \quad \text{for all } n.$$

Now we start with lemmas needed to prove the theorem quoted in Section 0. Our proof splits into several cases.

2. Ternary polynomials. In this section we deal with ternary polynomials over a special kind of binary algebras.

LEMMA 2.1. *Let $+$ and $\cdot$ be essentially binary polynomials over an algebra $\mathfrak{A}$. Assume that $+$ is idempotent and commutative. Then the polynomial*

$$q(x_1, x_2, x_3, x_4) = x_1 x_2 + x_3 x_4$$

is essentially 4-ary over $\mathfrak{A}$.

Proof. Since $+$ is idempotent, we get

$$xy = q(x, y, x, y).$$

Hence q is not constant. Assume now that q does not depend on x_i ($i = 1, 2, 3, 4$), say, x_1. Then using the identity

$$q(x_1, x_2, x_3, x_4) = q(x_3, x_4, x_1, x_2)$$

we infer that q also does not depend on x_3. This proves that the polynomial $q(x, y, x, y) = xy$ does not depend on x, which contradicts the fact that xy is essentially binary.

LEMMA 2.2. *Under the assumptions as above the polynomials*

$$f_0(x, y, z) = (x+y)+z, \quad f_1(x, y, z) = (x+y)z, \tag{1}$$
$$f_2(x, y, z) = z(x+y) \quad \text{and} \quad f_3(x, y, z) = xy+z$$

are essentially ternary.

Proof. It is clear that the polynomials f_0, f_1, f_2 admit the transposition (x, y) of their variables. We also have

$$f(x, x, y) \in \{x+y, xy, yx\} \qquad \text{for } f \in \{f_0, f_1, f_2\}.$$

Now using the commutativity and the idempotency of $x+y$ and the fact that both $x+y$ and xy are essentially binary we infer that f_i ($i = 0, 1, 2$) are essentially ternary. To prove that f_3 of (1) is essentially ternary it suffices to use Lemma 2.1 and the fact that

$$f_3(x_1, x_2, x_3 x_4) = q(x_1, x_2, x_3, x_4).$$

This completes the proof of the lemma.

LEMMA 2.3. *Let $(A, +, \cdot, \circ)$ be a proper idempotent algebra of type $(2, 2, 2)$ such that $(A, +, \cdot) \in B(+, \cdot)$ and $(A, \circ)$ is noncommutative. Then the polynomials*

$$\begin{gathered}(x+y)\circ z, \quad z\circ(x+y), \\ (xy)\circ z, \quad z\circ(xy), \\ (x+y)z, \quad xy+z\end{gathered} \tag{2}$$

admit only trivial permutations of their variables, i.e., the identity permutation and the transposition (x, y).

Proof. We give only the proof for the polynomial $(x+y)\circ z$ since the proof for the remaining polynomials of (2) runs similarly. It is clear that if $(x+y)\circ z$ admits a nontrivial permutation of its variables, then $(x+y)\circ z$ is a symmetric function. Hence

$$x+y = (x+y)\circ(x+y) = ((x+y)+y)\circ x = (x+y)\circ x = (x+x)\circ y = x\circ y,$$

which is impossible in a proper algebra $(A, +, \cdot, \circ)$.

LEMMA 2.4. *Under the assumptions of Lemma 2.3, the polynomials* $(x+y)\circ z$, $z\circ(x+y)$, $(x+y)z$ *and* $xy+z$ *are pairwise distinct (the same is true for the polynomials* $(xy)\circ z$, $z\circ(xy)$, $(x+y)z$ *and* $xy+z$*).*

Proof. Observe that every identity of the two above polynomials gives a contradiction. In fact, if, e.g., $(x+y)z = z\circ(x+y)$, then putting $x = y$ we get $yz = z\circ y$. This proves that $(A, \circ)$ is commutative, a contradiction.

LEMMA 2.5. *The assumption of Lemma 2.3 implies*

$$(x+y)\circ z \neq z\circ(xy) \quad \textit{and} \quad z\circ(x+y) \neq (xy)\circ z.$$

The proof is trivial by letting $x = y$.

LEMMA 2.6. *Under the assumptions of Lemma 2.3 the identities*

$$(x+y)\circ z = (xy)\circ z \quad \textit{and} \quad z\circ(x+y) = z\circ(xy)$$

do not hold simultaneously in the algebra $(A, +, \cdot, \circ)$.

Proof. In fact, if the converse holds, then

$$x+y = (x+y)\circ(x+y) = (xy)\circ(x+y) = (xy)\circ(xy) = xy,$$

a contradiction.

LEMMA 2.7. *There exists a proper idempotent algebra* $(A, +, \cdot, \circ)$ *of type* $(2, 2, 2)$ *such that* $(A, +, \cdot)$ *is a bisemilattice,* $(A, \circ)$ *is a noncommutative groupoid and* $(x+y)\circ z = (xy)\circ z$ *holds in the algebra* $(A, +, \cdot, \circ)$.

Proof. Take any nontrivial Płonka sum $(A, +, \cdot)$ of some proper lattices $(L_i, +, \cdot)_{i\in I}$ (for the definition of a Płonka sum of algebras see [18]). Then, of course, the polynomial $x\circ y = xy+y$ is essentially binary, idempotent and noncommutative. We also have $(A, +, \cdot) = (A, +, \cdot, \circ)$, since the sets of polynomials in both algebras $(A, +, \cdot)$ and $(A, +, \cdot, \circ)$ are equal. It is clear that $(A, +, \cdot, \circ)$ is a proper idempotent algebra of type (2, 2, 2) such that $(A, +, \cdot)$ is a bisemilattice and $(A, \circ)$ is noncommutative. Further, we have $(x+y)\circ z = (xy)\circ z$, since the identity $(x+y)z+z = (xy)z+z$ holds in every lattice, and therefore, as a regular identity, it also holds in the considered algebra (for details see [17] and [18]).

LEMMA 2.8. *Let* $(A, +, \cdot)$ *be a proper bisemilattice and let* $(A, \circ)$ *be a proper idempotent noncommutative groupoid. Then the inequalities*

$$(x+y)\circ z \neq (xy)\circ z \quad \textit{and} \quad z\circ(x+y) \neq z\circ(xy) \tag{3}$$

imply $p_3(A, +, \cdot, \circ) \geqslant 20$.

Proof. Consider a proper algebra $(A, +, \cdot, \circ)$ of type $(2, 2, 2)$. Applying Lemma 2.2 we infer that the polynomials $(x+y)+z$, $(x+y)z$, $xy+z$, $(xy)z$, $(x+y)\circ z$, $z\circ(x+y)$, $(xy)\circ z$ and $z\circ(xy)$ are essentially ternary. Now permuting variables in these polynomials and applying Lemmas 2.3–2.6 and the assumption (3) we get 20 different and essentially ternary polynomials over the considered algebra $(A, +, \cdot, \circ)$, namely: $x+y+z$, xyz, $(x+y)z$, $(y+z)x$, $(z+x)y$, $xy+z$, $yz+x$, $zx+y$, $(x+y)\circ z$, $(y+z)\circ x$, $(z+x)\circ y$, $z\circ(x+y)$, $y\circ(z+x)$, $x\circ(y+z)$, $(xy)\circ z$, $(yz)\circ x$, $(zx)\circ y$, $z\circ(xy)$, $y\circ(zx)$ and $x\circ(yz)$. The proof of the lemma is completed.

LEMMA 2.9. *Let $(A, +, \cdot, \circ, *)$ be a proper idempotent algebra of type $(2, 2, 2, 2)$ such that $(A, +, \cdot)\in B(+, \cdot)$, both groupoids $(A, \circ)$ and $(A, *)$ are noncommutative and $x\circ y\notin\{x*y, y*x\}$. Then*

$$p_3(A, +, \cdot, \circ, *)\geqslant 20. \tag{4}$$

Proof. Consider the following ternary polynomials over the algebra $(A, +, \cdot, \circ, *)$:

$$x+y+z,\quad xyz,\quad (x+y)z,\quad xy+z,\quad (x+y)\circ z,$$

$$z\circ(x+y),\quad (x+y)*z\quad \text{and}\quad z*(x+y).$$

Using Lemma 2.2 we infer that all these polynomials are essentially ternary. Now permuting variables in these polynomials and applying Lemmas 2.3–2.5 and the assumption that the polynomials $x\circ y$, $x*y$, $y*x$ are distinct we get (4).

3. Bisemilattices with one absorption law. In this section we deal with the variety of all bisemilattices $(A, +, \cdot)$ satisfying the absorption law $(x+y)y = y$. This variety will be denoted by $B_{a\cdot}(+, \cdot)$ or, shortly, by $B_{a\cdot}$ (we shall sometimes write B_a instead of $B_{a\cdot}$). The symbol $B_{a+}(+, \cdot)$ (or, shortly, B_{a+}) stands for the subvariety of $B(+, \cdot)$ of all bisemilattices $(A, +, \cdot)$ satisfying the identity $xy+y = y$ (for details see [8]).

Let n be a positive integer. Denote by $D(n)$ the set of all divisors of n. The symbols $\min(x, y)$, $\max(x, y)$, (x, y) and $[x, y]$, where $x, y\in N$, have their usual meanings. Now we have

LEMMA 3.1. *There exists a bisemilattice from the variety B_a being not a lattice.*

Proof. It is not difficult to check that the algebra $(D(n), +, \cdot)$, where $x+y = (x, y)$ and $xy = \max(x, y)$, belongs to the variety B_a for every n. We also see that this algebra does not satisfy the identity $xy+y = y$ for some n. For example, if $n = 12$, then $(D(12), +, \cdot)$ is not a lattice, since $xy+y\neq y$ for $x = 3$ and $y = 2$ (see also [5]).

LEMMA 3.2. *If $(A, +, \cdot)\in B_{a\cdot}$ and $(A, +, \cdot)$ is not a lattice, then the polynomial $x\circ y = xy+y$ is essentially binary and noncommutative (the dual version of this statement is also true, i.e., if $(A, +, \cdot)\in B_{a+}$ and $(A, +, \cdot)$ is not* *a lattice, then the polynomial* $(x+y)y$ *is essentially binary and noncommutative*).

Proof. First of all observe that $(A, +, \cdot)$ is proper. Indeed, if $\operatorname{card} A = 1$, then $(A, +, \cdot)$ as a one-element algebra is a lattice, a contradiction. Thus $\operatorname{card} A \geqslant 2$, which implies that both polynomials $x+y$ and xy are essentially binary. If $x+y = xy$, then $xy = (xy)y = (x+y)y = y$, a contradiction. So the algebra $(A, +, \cdot)$ is proper. Now applying Lemma 1 of [1] we infer that $x \circ y = xy + y \neq x$. Since $(A, +, \cdot)$ is not a lattice, $x \circ y \neq y$. Therefore, the polynomial $x \circ y$ is essentially binary. If $\circ$ is commutative, then using Lemma 2 of [1] we obtain $x \circ y = xy$, and hence

$$xy = (xy)y = (x \circ y)y = (xy+y)y = y,$$

a contradiction. Analogously one can prove the dual version of the lemma.

LEMMA 3.3. *If* $(A, +, \cdot) \in B_a$ *and* $(A, +, \cdot)$ *is not a lattice, then*

$$(x+y) \circ z \neq (xy) \circ z \quad \textit{and} \quad z \circ (x+y) \neq z \circ (xy),$$

where $x \circ y = xy + y$ (*the dual version of this lemma is also true*).

Proof. If $(x+y) \circ z = (xy) \circ z$, then $(x+y)z + z = xyz + z$. Putting $y = z$ in the last identity, we get

$$y = y + y = (x+y)y + y = xyy + y = xy + y.$$

This proves that $(A, +, \cdot)$ is a lattice, which is impossible. If again $z \circ (x+y) = z \circ (xy)$ holds, then

$$\begin{aligned} x+y &= (x+y) \circ (x+y) = (x+y) \circ (xy) = (x+y)(xy) + xy \\ &= ((x+y)y)x + xy = xy + xy = xy, \end{aligned}$$

which gives $xy = (xy)y = (x+y)y = y$, a contradiction.

LEMMA 3.4. *If* $(A, +, \cdot)$ *is a bisemilattice being not a lattice satisfying one absorption law, then* $p_3(A, +, \cdot) \geqslant 20$.

Proof. We give only the proof for a bisemilattice $(A, +, \cdot)$ from $B_{a \cdot}$ (the proof runs analogously for algebras from the variety B_{a+}). It is routine to prove that $(A, +, \cdot) \in B_{a \cdot}$ is proper if and only if $\operatorname{card} A \geqslant 2$. Now using Lemma 3.2 we infer that $(A, +, \cdot, \circ)$ is a proper idempotent algebra of type (2, 2, 2) such that $(A, \circ)$ is noncommutative. Applying Lemmas 2.8 and 3.3 we get the assertion.

COROLLARY 3.1. *There is no bisemilattice with one absorption law being not a lattice having* 19 *essentially ternary polynomials over it.*

Proof. Using Lemma 3.1 we deduce that there exist bisemilattices $(A, +, \cdot)$ with one absorption law which are not lattices. Applying Lemma 3.4 for such bisemilattices we get $p_3 \geqslant 20$. This completes the proof of the corollary.

4. Birkhoff's systems. Recall that a bisemilattice $(A, +, \cdot)$ is called a *Birkhoff system* if $(x+y)y = xy+y$ holds for $x, y \in A$ (see [1] and [16]). Denote by $B_b(+, \cdot)$ or, shortly, by B_b the variety of all Birkhoff's systems. It is clear that every lattice is a Birkhoff system. However, the converse is not true. Take, e.g., any nontrivial Płonka sum of some lattices. Then this algebra satisfies $(x+y)y = xy+y$ since the Płonka sum of algebras preserves regular identities (see [17] and [18]). It is also clear that the considered algebra is not a lattice. Some examples of Birkhoff's systems can be found in [8], [9] and [16].

LEMMA 4.1. *If $(A, +, \cdot) \in B_b$, then $(A, +, \cdot)$ satisfies the following identities*:

$$(xy+y)x = (xy+y)(yx+x) = xy,$$

$$(x+y)y+x = x+y+xy = x+y, \quad (xy+y)(x+y) = xy+y.$$

The proof can be found in [8].

LEMMA 4.2. *If $(A, +, \cdot)$ is proper, $(A, +, \cdot) \in B_b$ and $(A, +, \cdot)$ is not a lattice, then $x \circ y = (x+y)y = xy+y$ is essentially binary and noncommutative.*

The lemma follows from Theorem 1 and Lemma 3 of [3].

LEMMA 4.3. *If $(A, +, \cdot)$ is proper and $(A, +, \cdot) \in B_b$, then*

$$z \circ (x+y) \neq z \circ (xy), \quad \text{where } x \circ y = xy+y.$$

Proof. In fact, if $z \circ (x+y) = z \circ (xy)$, then putting $x = z$, we get

$$\begin{aligned} xy &= xy+xy = x(xy)+xy = x \circ (xy) \\ &= x \circ (x+y) = x(x+y)+x+y = x+y+xy. \end{aligned}$$

Hence $xy = x+y+xy$. Applying Lemma 4.1 we get $x+y = xy$, which is impossible in a proper bisemilattice.

LEMMA 4.4. *If $(A, +, \cdot)$ is a proper Birkhoff system, then the polynomials $xz+yz$ and $(x+z)(y+z)$ admit only trivial permutations of its variables (i.e., they admit only the transposition (x, y) and the identity permutation).*

Proof. We give only the proof for the polynomial $xz+yz$. It is clear that if $xz+yz$ admits a nontrivial permutation of its variables, then $xz+yz$ is symmetric, and hence $xz+yz = xz+xy$. Putting $y = z$ in this identity, we get

$$xy = xy+xy = xy+yy = xy+y.$$

Thus the polynomial $(x+y)y$ is commutative. Applying Lemma 2 of [3] we obtain $x+y = xy$, a contradiction.

LEMMA 4.5. *If $(A, +, \cdot)$ is a proper algebra from $V(+, \cdot)$, then*

$$xz+yz \neq (x+z)(y+z).$$

Proof. If $xz+yz = (x+z)(y+z)$ holds, then putting $x = y$ we get

$$yz = yz+yz = (y+z)(y+z) = y+z,$$

a contradiction.

Further we shall consider the following polynomials over Birkhoff's systems:

$$\begin{gathered} x+y+z, \quad xyz, \quad (x+y)z, \quad xy+z, \\ (x+y)\circ z, \quad z\circ(x+y), \quad (xy)\circ z, \quad z\circ(xy), \\ xz+yz, \quad (x+z)(y+z), \quad xy+yz+zx \quad \text{and} \quad (x+y)(y+z)(z+x), \end{gathered} \tag{5}$$

where $x\circ y = xy+y$.

LEMMA 4.6. *If $(A, +, \cdot)$ is a proper bisemilattice, then the polynomials*

$$q(x, y, z) = xz+yz \quad \textit{and} \quad \hat{q}(x, y, z) = (x+z)(y+z)$$

are essentially ternary (the same is true for any non-one-element algebra from $V(+, \cdot)$).

Proof. The assertion follows from the identities

$$q(x, x, y) = xy, \quad \hat{q}(x, x, y) = x+y,$$

$$q(x, y, z) = q(y, x, z) \quad \text{and} \quad \hat{q}(x, y, z) = q(y, x, z).$$

LEMMA 4.7. *If $(A, +, \cdot)$ is a proper Birkhoff system being not a lattice, then all the polynomials of* (5) *are essentially ternary and each of the polynomials $(x+y)z$, $xy+z$, $(x+y)\circ z$, $z\circ(x+y)$, $(xy)\circ z$, $z\circ(xy)$, $xz+yz$ and $(x+z)(y+z)$ admits only the identity permutation and the transposition (x, y) of its variables.*

Proof. Using Lemmas 2.2, 4.2, 4.6 and the fact that the two last polynomials of (5) are symmetric we infer that all polynomials of (5) are essentially ternary. Now applying Lemmas 2.3, 4.2 and 4.4 we deduce that all the polynomials listed in the lemma do not admit any nontrivial permutation of their variables. This completes the proof of the lemma.

LEMMA 4.8. *If $(A, +, \cdot)$ is a proper Birkhoff system, then*

$$x+y+z \neq xy+yz+zx \quad \textit{and} \quad xyz \neq xy+yz+zx.$$

Proof. If $x+y+z = xy+yz+zx$, then

$$x+y = x+y+y = xy+yx+x = xy+y.$$

Now an application of Lemma 3 of [3] gives $x+y = xy$, which is impossible in a proper bisemilattice. Analogously one proves the second inequality of the lemma.

Recall that a bisemilattice $(A, +, \cdot)$ (in general an algebra $(A, +, \cdot)$ of type (2, 2)) is called *distributive* if $(x+y)z = xz+yz$ and $xy+z = (x+z)(y+z)$ hold in $(A, +, \cdot)$ for all $x, y, z \in A$.

LEMMA 4.9. *Let $(A, +, \cdot)$ be a nondistributive Birkhoff system being not a lattice. Then $p_3(A, +, \cdot) \geqslant 21$.*

Proof. Since $(A, +, \cdot)$ is nondistributive, we have

$$(x+y)z \neq xz+yz \quad \text{or} \quad xy+z \neq (x+z)(y+z).$$

Without loss of generality we may assume that the first inequality holds. In virtue of Lemma 4.2 we infer that $x \circ y = xy+y$ is essentially binary and noncommutative. Consider now the following polynomials: $(x+y)z$, $xy+z$, $(x+y)\circ z$, $z\circ(x+y)$, $z\circ(xy)$ and $xz+yz$. By Lemma 4.7 we deduce that all these polynomials are essentially ternary and that the admissible group of each of them is isomorphic to S_2. If $xy+z = xy+yz$, then $x+y = xx+y = xy+xy = xy$, a contradiction. Now applying Lemmas 2.4, 4.2, 4.3 and the assumption $(x+y)z \neq xz+yz$ we conclude that all mentioned ternary polynomials are pairwise distinct. Permuting variables in these polynomials we get 18 essentially ternary and pairwise distinct polynomials. Adding to these 18 polynomials the polynomials $x+y+z$, xyz and $xy+yz+zx$ and applying Lemmas 4.6 and 4.8 we get 21 essentially ternary polynomials over $(A, +, \cdot)$. The proof of the lemma is completed.

LEMMA 4.10. *If $(A, +, \cdot)$ is a proper Birkhoff system being not a lattice such that $p_3(A, +, \cdot) < 21$, then $(A, +, \cdot)$ is distributive.*

Proof. We should add that there exist algebras satisfying the assumption of this lemma (see the beginning of this section and the next lemma). The lemma follows from Lemma 4.9.

LEMMA 4.11. *If $(A, +, \cdot)$ is a proper distributive bisemilattice being not a lattice, then $p_3(A, +, \cdot) = 18$.*

Proof. First of all observe that such a bisemilattice is a Birkhoff system, and it is also a distributive quasilattice considered by Płonka in [17]. For such algebras the polynomial $x \circ y = xy+y$ is a P-function (see [18]). Using the identities of the P-function (see [18]) and the Marczewski formula of the description of the set $A^{(n)}(\mathfrak{A})$ for a given algebra $\mathfrak{A}$ (see [15]) we infer that $x+y+z$, $(x+y)z$, $(y+z)x$, $(z+x)y$, xyz, $xy+z$, $yz+x$, $zx+y$, $xy+yz+zx$, $(x+y)\circ z$, $(y+z)\circ x$, $(z+x)\circ y$, $z\circ(x+y)$, $x\circ(y+z)$, $y\circ(z+x)$, $z\circ(xy)$, $x\circ(yz)$ and $y\circ(xz)$ are the only essentially ternary polynomials over $(A, +, \cdot)$. In the proof we have also used Lemmas 4.7 and 4.8.

COROLLARY 4.1. *There is no Birkhoff system $(A, +, \cdot)$ being not a lattice for which $p_3(A, +, \cdot) = 19$.*

Proof. Assume that such an algebra, say, $(A_0, +, \cdot)$, exists. Since $p_3(A_0, +, \cdot) = 19 < 21$, we infer that $(A_0, +, \cdot)$ is distributive (see Lemma 4.10). Applying now Lemma 4.11 we get $p_3(A_0, +, \cdot) = 18$, which is impossible.

5. Bisemilattices with $(x+y)y = x+y$. In this section we shall consider the variety $B_{c\cdot}(+,\cdot)$ or, shortly, $B_{c\cdot}$ (or B_c) of all bisemilattices satisfying the identity $(x+y)y = x+y$. Using Lemma 2 of [3] we see that it is the same to assume that the polynomial $(x+y)y$ is commutative, i.e., $(x+y)y = (y+x)x$. This identity (together with Lemma 2 of [3]) justifies the symbol $B_{c\cdot}(+,\cdot)$ for the above variety. Analogously, by $B_{c+}(+,\cdot)$ we denote the subvariety of all bisemilattices $(A, +, \cdot)$ satisfying the identity $xy+y = xy$.

Lemma 5.1. *There exist proper algebras in $B_{c\cdot}$.*

Proof. It suffices to take an algebra $(D(n), [x, y], \max(x, y))$ (see Section 3) for n being not a prime power, i.e., n is not of the form p^m, where p is a prime number and m is a positive integer (for details see [5]).

Lemma 5.2. *If $(A, +, \cdot)$ is proper and $(A, +, \cdot) \in B_{c\cdot}$, then the polynomial $x \circ y = xy+y$ is essentially binary and noncommutative (the dual version of the lemma is also true).*

Proof. Applying Lemma 1 of [3] we infer that $x \circ y \neq x$. If $x \circ y = y$, then
$$xy = y(xy) = (x \circ y)(xy) = (y+xy)(xy) = y+xy = xy+y = y,$$
a contradiction since $(A, +, \cdot)$ is a proper algebra. If $x \circ y$ is commutative, then $xy+y$ and $(x+y)y$ are both commutative, which contradicts Lemma 3 of [3].

Lemma 5.3. *If $(A, +, \cdot)$ is a proper algebra from $B_{c\cdot}$, then*
$$(x+y)\circ z \neq (xy)\circ z \quad \text{and} \quad z\circ(x+y) \neq z\circ(xy),$$
where $x \circ y = xy+y$.

Proof. Assume that $(x+y)\circ z = (xy)\circ z$ holds in $(A, +, \cdot)$. Then
$$\begin{aligned} x+y &= (x+y)+y = (x+y)y+y \\ &= (x+y)\circ y = (xy)y+y = xy+y = x\circ y. \end{aligned}$$
Thus $x \circ y$ is commutative, which contradicts Lemma 5.2. If again $z\circ(x+y) = z\circ(xy)$ holds in the algebra, then
$$\begin{aligned} xy &= xy+xy = x(xy)+xy = x\circ(xy) \\ &= x\circ(x+y) = x(x+y)+(x+y) = x+y. \end{aligned}$$
Hence $x+y = xy$, which is impossible.

Lemma 5.4. *If $(A, +, \cdot)$ is a proper algebra from $B_{c\cdot}$, then $p_3(A, +, \cdot) \geqslant 20$.*

This follows from Lemmas 2.8, 5.2 and 5.3.

From Lemma 5.4 we get immediately

Corollary 5.1. *There is no bisemilattice $(A, +, \cdot)$ in $B_{c\cdot}$ for which $p_3(A, +, \cdot) = 19$ (the same is true for the variety B_{c+}).*

6. Bisemilattices with at least five essentially binary polynomials. The main aim of this section is to prove Corollary 6.1 which states that there is no bisemilattice $(A, +, \cdot)$ for which $p_3(A, +, \cdot) = 19$ and $p_2(A, +, \cdot) \geqslant 5$. Before proving this we need three lemmas.

LEMMA 6.1. *If $(A, +, \cdot)$ is a proper bisemilattice, then*

$$(x+y)y \neq yx+x.$$

Proof. If $(x+y)y = yx+x$ holds in $(A, +, \cdot)$, then

$$\begin{aligned}(x+y)y &= ((x+y)+y)y = y(x+y)+(x+y) = (x+y)y+(x+y)\\ &= (yx+x)+(x+y) = (xy)+(x+y).\end{aligned}$$

Thus both polynomials $(x+y)y$ and $xy+y$ are commutative, which contradicts Lemma 3 of [3].

LEMMA 6.2. *If $(A, +, \cdot)$ is a bisemilattice and $p_2(A, +, \cdot) \geqslant 5$, then $p_3(A, +, \cdot) \geqslant 20$.*

Proof. Consider the following two binary polynomials:

$$x \circ y = (x+y)y \quad \text{and} \quad x * y = xy+y.$$

If both polynomials $x \circ y$ and $x * y$ are essentially binary, noncommutative and $x \circ y \neq x * y$, then using Lemmas 2.9 and 6.1 we infer that

$$p_3(A, +, \cdot) = p_3(A, +, \cdot, \circ, *) \geqslant 20.$$

If $x \circ y = x * y$, then $(A, +, \cdot)$ is a Birkhoff system. Now using Lemma 4.1 we deduce that $p_2(A, +, \cdot) \leqslant 4$, a contradiction with the assumption (see also [8]). Applying Lemma 1 of [3] we have $(x+y)y \neq x$ and $xy+y \neq x$. If $x \circ y$ is not essentially binary, then $x \circ y = y$. If also $x * y = y$, then $(A, +, \cdot)$ is a lattice, and therefore $p_2(A, +, \cdot) \leqslant 2$, which contradicts the assumption $p_2(A, +, \cdot) \geqslant 5$. Thus further we have to consider the case where $(A, +, \cdot) \in B_{a\cdot}$ and $(A, +, \cdot)$ is not a lattice or the dual case where $(A, +, \cdot) \in B_{a+}$ and $(A, +, \cdot)$ is not a lattice. Applying now Lemma 3.4 we get $p_3(A, +, \cdot) \geqslant 20$. It remains to consider the case where $(x+y)y$ is essentially binary and commutative (or the dual one). By Lemma 2 of [3] we infer that $(A, +, \cdot) \in B_{c\cdot}$. Since $p_2(A, +, \cdot) \geqslant 5$, we deduce that $(A, +, \cdot)$ is proper. Applying now Lemma 5.4 we get $p_3(A, +, \cdot) \geqslant 20$. If $x * y = xy+y$ is commutative, then we use the dual versions of Lemma 2 of [3] and of Lemma 5.4. The proof of the lemma is completed.

LEMMA 6.3. *If $(A, +, \cdot)$ is a proper bisemilattice being not a lattice, then $p_3(A, +, \cdot) = 18$ or $p_3(A, +, \cdot) \geqslant 20$.*

Proof. Since $(A, +, \cdot)$ is not a lattice, we infer by Theorem 1 of [3] that $p_2(A, +, \cdot) > 2$. Using Theorem 2 of [3] we obtain $p_2(A, +, \cdot) \geqslant 4$. If $p_2(A, +, \cdot) \geqslant 5$, then applying Lemma 6.2 we get $p_3(A, +, \cdot) \geqslant 20$. Take now into account the case $p_2(A, +, \cdot) = 4$. Considering as in the proof of Lemma 6.2 the polynomials $x \circ y = (x+y)y$ and $x * y = xy+y$ we infer that under the condition $p_2(A, +, \cdot) = 4$ the algebra $(A, +, \cdot)$ is a Birkhoff system being not a lattice. Thus $(A, +, \cdot)$ is in the variety $B_b(+, \cdot)$. If $(A, +, \cdot)$ is nondistributive, then applying Lemma 4.9 we get $p_3(A, +, \cdot) \geqslant 21 > 20$. If $(A, +, \cdot)$ is a proper distributive bisemilattice being not a lattice, then we use Lemma 4.11 to get the requirement. This completes the proof of the lemma.

From Lemma 6.3 we get

COROLLARY 6.1. *There is no bisemilattice $(A, +, \cdot)$ with $p_2(A, +, \cdot) \geqslant 5$ satisfying $p_3(A, +, \cdot) = 19$.*

We should mention here that there exist bisemilattices $(A, +, \cdot)$ for which $p_2(A, +, \cdot) \geqslant 5$. Such bisemilattices are considered in [9].

7. Ternary polynomials in lattices. In this section we present some lemmas on ternary polynomials in lattices. We also give some characterizations for distributive lattices and nondistributive modular lattices.

LEMMA 7.1. *If $(L, +, \cdot)$ is a lattice with* card $L \geqslant 2$, *then the polynomials*

$$(6)\qquad \begin{aligned} a &= x+y+z, & \hat{a} &= xyz, \\ b &= (x+y)z, & \hat{b} &= xy+z, \\ c &= xz+yz, & \hat{c} &= (x+z)(y+z), \\ d &= xy+yz+zx, & \hat{d} &= (x+y)(y+z)(z+x), \\ e &= (x+y)(xy+z), & \hat{e} &= xy+(x+y)z, \\ f &= (xy+z)x, & \hat{f} &= (x+y)z+x \end{aligned}$$

are essentially ternary.

Proof. For a, $\hat{a}$, b and $\hat{b}$ the statement follows from Lemma 2.2. The polynomials d and $\hat{d}$ are essentially ternary, since they are idempotent and symmetric. Using Lemma 4.6 we infer that c and $\hat{c}$ are essentially ternary. Now we prove that e is essentially ternary (the proof goes analogously for $\hat{e}$). Observe that $e(x, y, z) = e(y, x, z)$ and $e(y, y, z) = y$. This proves that e depends on x and y. If e does not depend on z, then $e(x, y, z) = e(x, y, y)$, whence $x = y$, a contradiction. Consider now the polynomial $f(x, y, z) = (xy+z)x$. We have $xz = (f(x, y, z))z$, which proves that f depends on x. If f does not depend on z, then $(xy+z)x = (xy+y)x$. Hence $xy = x$, a contradiction. If f does not depend on y, then $f(x, y, z) = f(x, x, z) = x$. This proves that f does not depend on the variable z, a contradiction. Analogously one can prove that $\hat{f}$ is not essentially ternary.

LEMMA 7.2. *If $(L, +, \cdot)$ is a lattice with* card $L \geqslant 2$, *then the polynomials f*

and $\hat{f}$ of (6) neither admit any cycles nor any transpositions (x, y) and (x, z) of their variables.

Proof. We give only the proof for the polynomial f (the proof for $\hat{f}$ runs similarly). If f admits a cycle of its variables x, y, z, then

$$f(x, y, z) = (f(x, y, z))x = (f(y, z, x))x = ((yz+x)y)x = ((yz+x)x)y = xy.$$

Hence $f(x, y, z) = xy$, which contradicts the previous lemma. The proof runs similarly if f admits the transpositions (x, z). Let now f admit the transposition (x, y). Then

$$xz = (f(x, y, z))z = (f(y, x, z))z = ((yx+z)y)z = yz.$$

Hence $xy = x$, a contradiction.

LEMMA 7.3. *Let $(L, +, \cdot)$ be a lattice. Then $(L, +, \cdot)$ is modular if and only if $(xy+z)x = (xz+y)x$ holds in $(L, +, \cdot)$ (the dual version of the lemma is also true).*

Proof. Assume that $(L, +, \cdot)$ is a modular lattice. Then $(L, +, \cdot)$ satisfies $(xy+z)x = xy+xz$. Hence $(xy+z)x = (xz+y)x$ since the polynomial $xy+xz$ admits the transposition (y, z) of its variables. Assume now that $(L, +, \cdot)$ is a lattice satisfying $(xy+z)x = (xz+y)x$ and assume to the contrary that $(L, +, \cdot)$ is nonmodular. Using statement (i) of Theorem 2 (p. 70) of [12] we infer that L contains isomorphically $N_5 = \{0, a, b, c, 1\}$ as a sublattice, where $b < a$. Hence

$$a = 1 \cdot a = (b+c)a = (ab+c)a = (ac+b)a = (0+b)a = b,$$

a contradiction. Analogously one can prove the dual version of the lemma.

LEMMA 7.7. *If $(L, +, \cdot)$ is a nonmodular lattice, then the polynomials $f(x, y, z) = (xy+z)x$ (or its dual $\hat{f}$) does not admit any permutation of its variables, i.e., $G(f) \cong S_1$.*

Proof. If $(L, +, \cdot)$ is a nonmodular lattice, then the lemma follows from Lemmas 7.2 and 7.3. Assume now that $G(f) \cong S_1$ (analogously if $G(\hat{f}) \cong S_1$). Then $(L, +, \cdot)$ is nonmodular since in the opposite case the polynomial $f(x, y, z) = (xy+z)x = xy+xz$ admits the transposition (y, z) of its variables. This contradicts the assumption $G(f) \cong S_1$.

LEMMA 7.5. *If $(L, +, \cdot)$ is a nondistributive lattice, then the polynomials b, $\hat{b}$, c, $\hat{c}$ of (6) are pairwise distinct.*

The lemma follows from the nondistributivity of the lattice $(L, +, \cdot)$ and the method of proving used in Lemma 2.4.

LEMMA 7.6. *If $(L, +, \cdot)$ is a lattice with* card $L \geq 2$, *then the polynomials b, $\hat{b}$, c, $\hat{c}$ of (6) admit only the transposition (x, y) and the identity permutation of their variables.*

Proof. Applying Lemma 2.3 we get the required assertion for b and $\hat{b}$.

It is clear that c admits a nontrivial permutation of its variables; then c is symmetric, and hence $xz+yz = xz+xy$. Putting $y = z$ in this identity, we get

$$xy = xy+xy = xy+y = y,$$

a contradiction. The proof runs similarly for $\hat{c}$.

LEMMA 7.7. *If $(L, +, \cdot)$ is a nonmodular lattice, then the polynomials*

$$a = x+y+z, \quad \hat{a} = xyz,$$

$$b = (x+y)z,\ (y+z)x,\ (z+x)y,$$

$$\hat{b} = xy+z,\ yz+x,\ zx+y,$$

$$c = xz+yz,\ xy+zy,\ zx+yx,$$

$$\hat{c} = (x+z)(y+z),\ (x+y)(z+y),\ (z+x)(y+x),$$

$$d = xy+yz+zx, \quad \hat{d} = (x+y)(y+z)(z+x),$$

$$f = (xy+z)x,\ (yz+x)y,\ (zx+y)z,\ (yx+z)y,\ (xz+y)x,\ (zy+x)z$$

are all essentially ternary and pairwise distinct.

Proof. Using Lemma 7.1 we infer that all the above polynomials are essentially ternary. It is clear that $G(a) = G(\hat{a}) = G(d) = G(\hat{d}) \cong S_3$. Applying Lemmas 7.4 and 7.6 we obtain $G(f) \cong S_1$, $G(b) = G(\hat{b}) = G(c) = G(\hat{c}) \cong S_2$. Now using these facts, Lemma 7.5 and the nondistributivity of $(L, +, \cdot)$ (see Theorem 3.1, p. 19, of [2]) we get our requirement. The proof of the lemma is completed.

From Lemma 7.7 we get immediately

COROLLARY 7.1. *There is no nonmodular lattice $(L, +, \cdot)$ for which $p_3(L, +, \cdot) = 19$. Moreover, if $(L, +, \cdot)$ is a nonmodular lattice, then $p_3(L, +, \cdot) \geqslant 22$.*

8. Nondistributive modular lattices. In this section we give some more information about ternary polynomials over modular (distributive) lattices.

LEMMA 8.1. *Let $(L, +, \cdot)$ be a lattice. Then the following are equivalent:*

(i_1) *$(L, +, \cdot)$ is distributive;*

(i_2) *$(L, +, \cdot)$ satisfies $(x+y)(xy+z) = xy+yz+zx$;*

(i_3) *$(L, +, \cdot)$ satisfies $xy+(x+y)z = (x+y)(y+z)(z+x)$.*

Proof. The implications $(i_1) \Rightarrow (i_2)$ and $(i_1) \Rightarrow (i_3)$ are easy to verify. Now we prove $(i_2) \Rightarrow (i_1)$ (analogously one can prove the implication $(i_3) \Rightarrow (i_1)$). Putting xy for y in the identity

$$(x+y)(xy+z) = xy+yz+zx$$

we get

$$(xy+z)x = (x+xy)\big(x(xy)+z\big) = xy+xyz+zx = xy+xz.$$

Hence $xy+xz = x(xy+z)$, which proves that the considered lattice is modular. Now we have

$$\begin{aligned}(x+y)z &= (x+y)\big((xy+z)z\big)\\ &= \big((x+y)(xy+z)\big)z = (xy+yz+zx)z = \big((zx)+(xy+yz)\big)z\\ &= zx+(xy+yz)z = zx+(zy+xy)z = zx+zy+xyz = xz+yz.\end{aligned}$$

Thus $(L, +, \cdot)$ satisfies the distributivity.

LEMMA 8.2. *If $(L, +, \cdot)$ is a nondistributive modular lattice, then $d \neq \hat{e}$ and $e \neq \hat{d}$.*

Proof. Let $e = \hat{d}$, i.e.,

$$(x+y)(xy+z) = (x+y)(y+z)(z+x)$$

holds in $(L, +, \cdot)$. Using this identity we get

$$(xy+z)x = \big((x+y)(xy+z)\big)x = \big((x+y)(y+z)(z+x)\big)x = (y+z)x.$$

Since $(L, +, \cdot)$ is modular, we infer that

$$(y+z)x = (xy+z)x = yx+zx,$$

which proves that $(L, +, \cdot)$ is distributive, a contradiction. Analogously, using the dual modular law $(x+y)z+x = (x+y)(x+z)$, we prove that $\hat{e}$ and d are distinct.

LEMMA 8.3. *Let $(L, +, \cdot)$ be a modular lattice. Then the following are equivalent*:

(j_1) *$(L, +, \cdot)$ is nondistributive*;

(j_2) *the polynomial $e(x, y, z) = (x+y)(xy+z)$ is nonsymmetric*;

(j_3) *the polynomial $\hat{e}(x, y, z) = xy+(x+y)z$ is nonsymmetric.*

Proof. Let $(L, +, \cdot)$ be modular and nondistributive. Assume that the polynomial $e(x, y, z)$ is symmetric. Then

$$(x+y)(xy+z) = (x+z)(xz+y).$$

Using this identity we get

$$\begin{aligned}(x+y)z &= (x+y)\big((xy+z)z\big) = \big((x+y)(xy+z)\big)z = \big((x+z)(xz+y)\big)z\\ &= \big((x+z)z\big)(xz+y) = (zx+y)z.\end{aligned}$$

Hence $(x+y)z = (zx+y)z = xz+yz$, which proves that $(L, +, \cdot)$ is distributive, a contradiction. To prove that (j_2) $\Rightarrow$ (j_1) assume to the contrary that $(L, +, \cdot)$ is distributive. Then we have

$$e(x, y, z) = (x+y)(xy+z) = (x+y)(xy)+(x+y)z = xy+yz+zx.$$

Thus $e(x, y, z)$ is symmetric, a contradiction. Analogously we prove (j_1) $\Leftrightarrow$ (j_3). The proof of the lemma is completed.

LEMMA 8.4. *If* $(L, +, \cdot)$ *is a modular nondistributive lattice, then the following polynomials of* (6): a, $\hat{a}$, b, $\hat{b}$, c, $\hat{c}$, d, $\hat{d}$ *and* $e = \hat{e}$ *are pairwise distinct.*

Proof. The assertion follows from the nondistributivity of $(L, +, \cdot)$ (see Theorem 3.1, p. 19, of [2]), Lemmas 8.1, 8.2 and some standard methods of proving used earlier. For example, if

$$(x+y)(xy+z) = xy+yz+zx,$$

then using the modular law (see [2], p. 19) we infer that

$$xy+yz+zx = xy+(x+y)z.$$

Hence $d = \hat{e}$, which contradicts Lemma 8.2.

LEMMA 8.5. *If* $(L, +, \cdot)$ *is a modular nondistributive lattice, then the ternary polynomials*

$$\begin{gathered} a = x+y+z, \quad \hat{a} = xyz, \\ b = (x+y)z,\ (y+z)x,\ (z+x)y, \\ \hat{b} = xy+z,\ yz+x,\ zx+y, \\ c = xz+yz,\ xy+zy,\ zx+yx, \\ \hat{c} = (x+z)(y+z),\ (x+y)(z+y),\ (z+x)(y+x), \\ d = xy+yz+zx, \quad \hat{d} = (x+y)(y+z)(z+x), \\ e = (x+y)(xy+z),\ (y+z)(yz+x),\ (z+x)(zx+y) \end{gathered} \tag{7}$$

are essentially ternary and pairwise distinct.

Proof. The fact that the polynomials of (7) are essentially ternary follows from Lemma 7.1. Further the assertion follows from Lemmas 7.5, 7.6 and 8.2–8.4.

LEMMA 8.6. *If* $(L, +, \cdot)$ *is a modular nondistributive lattice, then the polynomials of* (7) *are the only essentially ternary polynomials over* $(L, +, \cdot)$.

Proof. To prove this lemma we use a description of the set $A^{(n)}(\mathfrak{A})$. Namely,

$$A^{(n)}(\mathfrak{A}) = \bigcup_{k=0}^{\infty} A_k^{(n)}(\mathfrak{A}),$$

where

$$\mathfrak{A} = (A, F), \quad A_0^{(n)}(\mathfrak{A}) = \{e_1^{(n)}, \ldots, e_n^{(n)}\}$$

and

$$A_{k+1}^{(n)}(\mathfrak{A}) = A_k^{(n)}(\mathfrak{A}) \cup \{f(f_1, \ldots, f_n): f_1, \ldots, f_m \in A_k^{(n)}(\mathfrak{A}) \text{ and } f \in F\}$$

$(k = 0, 1, \ldots)$. In our case $F = \{+, \cdot\}$ (for details see [15]). We should mention that we also use the identities of the variety of modular lattices, among others the identities

$$(xy+z)x = xy+xz, \qquad (x+y)z+x = (x+y)(x+z)$$

and

$$(x+y)(xy+z) = xy+(x+y)z$$

(see [2] and [12]).

LEMMA 8.7. *Let $(L, +, \cdot)$ be a nondistributive lattice. Then $(L, +, \cdot)$ is modular if and only if $p_3(L, +, \cdot) = 19$.*

Proof. If $(L, +, \cdot)$ is modular, then the assertion follows from Lemma 8.6. Assume now that $(L, +, \cdot)$ is a nondistributive lattice and $p_3(L, +, \cdot) = 19$ (in this case it suffices to assume that $(L, +, \cdot)$ is a lattice and $p_3(L, +, \cdot) = 19$, see [7]). If $(L, +, \cdot)$ is nonmodular, then applying Corollary 7.1 we get $p_3(L, +, \cdot) \geqslant 22$, a contradiction.

LEMMA 8.8. *Let $(L, +, \cdot)$ be a modular lattice. Then $(L, +, \cdot)$ is nondistributive if and only if $p_3(L, +, \cdot) = 19$.*

Proof. If $(L, +, \cdot)$ is nondistributive, then applying Lemmas 8.5 and 8.6 we get $p_3(L, +, \cdot) = 19$. If $p_3(L, +, \cdot) = 19$, then, of course, $(L, +, \cdot)$ is a nondistributive lattice, since in the opposite case we have $p_3(L, +, \cdot) = 9$ (if $\operatorname{card} L \geqslant 2$), which is impossible.

9. Proof of the Theorem. In this section we prove the theorem quoted in Section 0. Let $(A, +, \cdot)$ be a bisemilattice. If $(A, +, \cdot)$ is a modular nondistributive lattice, then the assertion follows from Lemma 8.7 or 8.8. Assume now that $p_3(A, +, \cdot) = 19$. This implies that $(A, +, \cdot)$ is a proper bisemilattice. Hence $p_2(A, +, \cdot) \geqslant 2$. Using Corollary 6.1 we infer that $p_2(A, +, \cdot) \leqslant 4$. Applying Theorem 2 of [3] we conclude that $p_2(A, +, \cdot) = 2$ or $p_2(A, +, \cdot) = 4$. If $p_2(A, +, \cdot) = 2$, then by Theorem 1 of [3] we deduce that $(A, +, \cdot)$ is a lattice. If now $(A, +, \cdot)$ is a nonmodular lattice, then applying Corollary 7.1 we get $p_3(A, +, \cdot) \geqslant 22$, a contradiction. Thus $(A, +, \cdot)$ is a modular lattice. Since $p_3(A, +, \cdot) = 19$, we infer by Lemma 8.8 that $(L, +, \cdot)$ is nondistributive. Let us turn to the remaining case where $p_2(A, +, \cdot) = 4$. All such bisemilattices are described in [8] and every bisemilattice $(A, +, \cdot)$ with $p_2(A, +, \ ) = 4$ belongs to one of the following varieties: $B_{a\cdot}(+, \cdot)$, $B_{a+}(+, \cdot)$, $B_b(+, \cdot)$, $B_{c\cdot}(+, \cdot)$ and $B_{c+}(+, \cdot)$. Now applying Corollaries 3.1, 4.1 and 5.1 we get a contradiction with the assumption $p_3(A, +, \cdot) = 19$. This completes the proof of the theorem.

REFERENCES

[1] G. Birkhoff, *Lattice Theory*, 2nd ed., Amer. Math. Soc., Providence, Rhode Island, 1948.
[2] P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, New York, 1973.
[3] J. Dudek, *On bisemilattices. I*, Colloq. Math. 47 (1982), pp. 1–5.
[4] – *On bisemilattices. II*, Demonstratio Math. 15 (1982), pp. 465–475.
[5] – *On bisemilattices. III*, Math. Sem. Notes Kobe. Univ. 10 (1982), pp. 275–279.
[6] – *On the variety* $V_\infty(+, \cdot)$, ibidem 10 (1982), pp. 9–15.
[7] – *A characterization of distributive lattices*, Colloq. Math. Soc. János Bolyai, 33. *Contributions to Lattice Theory*, Szeged (Hungary), 1980, North-Holland, Amsterdam 1983, pp. 325–335.
[8] – and A. Romanowska, *Bisemilattice with four essentially binary polynomials*, ibidem, pp. 337–359.
[9] J. Gałuszka, *Bisemilattices with five essentially binary polynomials*, Math. Slovaca (to appear).
[10] – *Generalized absorption laws in bisemilattices*, Algebra Universalis 19 (1984), pp. 304–318.
[11] G. Grätzer, *Universal Algebra*, Springer-Verlag, Berlin–Heidelberg–New York 1979.
[12] – *Lattice Theory. First Concepts and Distributive Lattices*, W. H. Freeman Co., San Francisco 1971.
[13] – and R. Padmanabhan, *On idempotent commutative and nonassociative groupoids*, Proc. Amer. Math. Soc. 28 (1971), pp. 75–80.
[14] G. Grätzer and J. Płonka, *On the number of polynomials of an idempotent algebra. I*, Pacific J. Math. 32 (1970), pp. 697–709.
[15] E. Marczewski, *Independence and homomorphisms in abstract algebras*, Fund. Math. 50 (1961), pp. 45–61.
[16] R. Padmanabhan, *Regular identities in lattices*, Trans. Amer. Math. Soc. 158 (1971), pp. 179–188.
[17] J. Płonka, *On distributive quasilattices*, Fund. Math. 60 (1967), pp. 197–200.
[18] – *On equational classes of abstract algebra defined by regular equations*, ibidem 64 (1969), pp. 241–247.
[19] W. Taylor, *Equational logic*, Houston J. Math., Survey (1979), pp. 1–83.

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