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CERTAIN CONNECTIONS BETWEEN TIME FUNCTIONS AND COMPACT SPACELIKE SUBMANIFOLDS

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0. Introduction. The present paper concerns $(n+1)$-dimensional Riemannian manifolds $(\bar{M}, g)$ with metric g of index one. A C^∞-function f on $\bar{M}$ is called a *time function* if its gradient df satisfies the relation $g(df, df) < 0$. In this case the quotient space $\bar{M}/f$ of $\bar{M}$ by the relation, whose equivalence classes are maximal integral curves of df, is an n-dimensional C^∞-manifold, in general non-Hausdorff.

The case where $\bar{M}/f$ is a Hausdorff manifold is considered in Section 2. It is shown there, in particular, that $\bar{M}$ is diffeomorphic to the product $\boldsymbol{R} \times \bar{M}/f$ (Theorem 1). Moreover, if a compact n-dimensional submanifold M of $\bar{M}$ is spacelike, that is $g(Y, Y) > 0$ whenever $0 \neq Y \in TM$, then M is diffeomorphic to $\bar{M}/f$ and meets each maximal integral curve of df at exactly one point. In particular, if $\bar{M}$ admits a time function f such that $\bar{M}/f$ is a Hausdorff space, then any two compact spacelike submanifolds of $\bar{M}$ are diffeomorphic (Corollary 1). Section 2 contains also a sufficient condition for a 1-form ω on $\bar{M}$ which is closed and timelike ($d\omega = 0$, $g(\omega, \omega) < 0$) to be the gradient of a time function.

The results of Section 3 show that the above-mentioned statements about time functions and compact spacelike submanifolds fail in the case where $\bar{M}/f$ is non-Hausdorff. Namely, each n-dimensional compact manifold can be embedded in some Riemannian manifold as a spacelike submanifold which meets an integral curve of the gradient of some time function at two points (Theorem 2). Theorem 3 states that for each $n \geqslant 1$ there exists an $(n+1)$-dimensional Riemannian manifold $(\bar{M}, g)$ with metric g of index one which admits a time function and contains a diffeomorphic image of each compact n-dimensional manifold as a spacelike submanifold. Therefore, the assumption that $\bar{M}/f$ is a Hausdorff space cannot be omitted in Corollary 1.

1. Some properties of time functions. By a *manifold* we shall mean a connected C^∞-manifold which need not be a Hausdorff space. In the sequel, $(\bar{M}, g)$ will denote an $(n+1)$-dimensional Hausdorff manifold $\bar{M}$ with Riemannian metric g of index one. A vector $Y \in T\bar{M}$ is called *spacelike* (respectively, *timelike*) if $g(Y, Y) > 0$ (respectively, $g(Y, Y) < 0$). An n-dimensional submanifold M of $\bar{M}$ is called *spacelike* if so is each non-zero vector tangent to M.

By a *time function* on $\bar{M}$ we shall mean a C^∞-function whose gradient is a timelike vector field. In the sequel, f will denote a time function on $\bar{M}$. By X we shall denote a complete timelike vector field on $\bar{M}$ such that X_p is parallel to df_p at each $p \in \bar{M}$. Such a field exists for each time function f. In fact, let h be any complete positive definite Riemannian metric on $\bar{M}$ which exists in view of Whitney's embedding theorem (see [3], p. 113). Then the field $X = h(df, df)^{-1/2} \cdot df$ is h-unit and therefore complete. The integral curves of X are the same as those of df, up to a change of parameter. The flow of X will be denoted by φ_t, $t \in \boldsymbol{R}$.

It is easy to verify the following

PROPOSITION 1. *Let f be a time function on $\bar{M}$. Then*

(i) *$\bar{M}$ is not compact.*

(ii) *If a curve $x: (a, b) \to \bar{M}$ is timelike, that is $g(\dot{x}_t, \dot{x}_t) < 0$ for $t \in (a, b)$, then f is strictly monotone along x. In particular, $\bar{M}$ admits no closed timelike curve.*

In fact, (i) is obvious and (ii) follows from the relation

$$\frac{d}{dt} f(x_t) = g(df, \dot{x}_t) \neq 0.$$

Given a time function f on $\bar{M}$, by $\bar{M}/f$ we shall denote the topological quotient space of $\bar{M}$ by the relation whose equivalence classes are maximal integral curves of df. The natural projection will be denoted by $\pi = \pi_f$: $\bar{M} \to \bar{M}/f$.

PROPOSITION 2. *The quotient $\bar{M}/f$ is an n-dimensional topological manifold (not necessarily a Hausdorff space). Moreover, it admits a unique differentiable structure of class C^∞ such that for any n-dimensional submanifold M of $\bar{M}$ which is transverse to df (in particular, for any spacelike submanifold) the mapping $\pi: M \to \bar{M}/f$ is locally diffeomorphic. The projection $\pi: \bar{M} \to \bar{M}/f$ is of class C^∞ with respect to this structure.*

Proof. Let $p \in \bar{M}$ and let M be any n-dimensional submanifold of $\bar{M}$, containing p and transverse to df. Since f is monotone decreasing along each integral curve of df, it is easy to see that for some neighbourhood U of p in M the projection $\pi: U \to \bar{M}/f$ is one-one. Let V be any open subset of U. Since the mapping

$$\boldsymbol{R} \times V \ni (t, q) \mapsto \varphi_t q \in \bar{M}$$

is locally diffeomorphic, its image

$$\bigcup_{t\in \boldsymbol{R}} \varphi_t V = \pi^{-1}(\pi(V))$$

is open in $\bar{M}$ and $\pi(V)$ is open in $\bar{M}/f$. Hence $\pi\colon U \to \bar{M}/f$ is a homeomorphic embedding and $\pi(U)$ is open. Therefore, the pair

$$(\pi(U), \psi) = (\pi(U), (\pi|U)^{-1})$$

is a local chart in $\bar{M}/f$.

Now it is sufficient to prove that the atlas on $\bar{M}/f$ formed by all charts of this kind is of class C^∞. Let $(\pi(U_i), \psi_i)$, $i = 1, 2$, be two such charts and let $p \in \psi_1(\pi(U_1)\cap\pi(U_2))$. Thus $p \in U_1$ and $\varphi_t p \in U_2$ for some real number t. Choose a coordinate system $x^0, \ldots, x^n$ at $\varphi_t p$ such that the equation $x^0 = 0$ defines a neighbourhood of $\varphi_t p$ in U_2. The function z, given by $z(s, q) = x^0(\varphi_s q)$, is defined in a neighbourhood of (t, p) in $\boldsymbol{R}\times U_1$ and satisfies the conditions

$$z(t, p) = 0 \quad \text{and} \quad \frac{\partial z}{\partial s}(t, p) = \frac{d}{ds} x^0(\varphi_s p)|_{s=t} = g(dx^0, X) \neq 0,$$

since dx^0 is orthogonal and X is transverse to U_2. Using the implicit function theorem we may find a C^∞-function G on a neighbourhood of p in U_1 such that $z(G(q), q) = 0$, i.e., $\varphi_{G(q)} q \in U_2$. Therefore, the transition mapping is given by $\psi_2\circ\psi_1^{-1}(q) = \varphi_{G(q)} q$, so it is of class C^∞.

Thus $\bar{M}/f$ is provided with a C^∞ differentiable structure. The assertion that π_f is of class C^∞ follows from the fact that df can be written locally in the form $\partial/\partial x^0$ for some coordinate system $x^0, \ldots, x^n$.

The particular case,

$$(\bar{M}, g) = (\boldsymbol{R}^2 - \{0\}, -(dx)^2 + (dy)^2), \quad f = x,$$

shows that the quotient cannot be a Hausdorff space. In fact, here it is diffeomorphic to the real line with one point doubled.

2. The Hausdorff case.

THEOREM 1. *Let $\bar{M}/f$ be a Hausdorff space. Then there exists a diffeomorphism Φ of the product $\boldsymbol{R}\times\bar{M}/f$ onto $\bar{M}$ such that, for any maximal integral curve L of df, $\Phi(\boldsymbol{R}\times\{L\}) = L$.*

Proof. Define $\varphi\colon \bar{M}\to\boldsymbol{R}\times\bar{M}/f$ by $\varphi(p) = (f(p), \pi(p))$. Using coordinate systems of the form $f, x^1, \ldots, x^n$ on $\bar{M}$ it is easy to verify that φ is locally diffeomorphic. Moreover, φ is one-one, since f decreases along each integral curve of df. Thus φ maps $\bar{M}$ diffeomorphically onto the open submanifold $\varphi(\bar{M})$ of $\boldsymbol{R}\times\bar{M}/f$.

We assert that there exists a C^∞-function $F\colon \bar{M}/f\to\boldsymbol{R}$ such that for each $L\in\bar{M}/f$ the real number $F(L)$ lies in the interval $f(L) = \{f(p) \mid p\in L\}$. It is clear that such a function exists in a neighbourhood of any given point of $\bar{M}/f$. Choose a locally finite open covering $\{U_i \mid i\in I\}$

of $\bar{M}/f$ with a family $\{F_i \mid i \in I\}$ of C^∞-functions $F_i\colon U_i \to \boldsymbol{R}$ such that $F_i(L) \in f(L)$, and with a partition of unity $\{G_i \mid i \in I\}$ such that $0 \leqslant G_i \leqslant 1$ and $\operatorname{supp} G_i \subset U_i$ for any $i \in I$. From the convexity of intervals it follows that the function

$$F = \sum_{i \in I} G_i F_i$$

has the properties stated above.

Now define $\Phi\colon \boldsymbol{R} \times \bar{M}/f \to \bar{M}$ by

$$\Phi(t, L) = \varphi_t\big(\varphi^{-1}(F(L), L)\big).$$

It is clear that the assignment $\bar{M}/f \ni L \mapsto \varphi^{-1}(F(L), L) \in \bar{M}$ defines a diffeomorphism of $\bar{M}/f$ onto a submanifold of $\bar{M}$ which is transverse to df and meets each maximal integral curve of df exactly once. Therefore, Φ is a diffeomorphism with the desired properties, which completes the proof.

To conclude some facts from Theorem 1 we shall need the following

LEMMA 1. *Suppose that N is an n-dimensional Hausdorff manifold and M is an n-dimensional compact submanifold of $\boldsymbol{R} \times N$, which is transverse to each curve of the form $\boldsymbol{R} \times \{p\}$, $p \in N$. Then M intersects any such curve at exactly one point, in particular M is diffeomorphic to N.*

Proof. Since M is compact, we may choose $\varepsilon > 0$ such that the conditions $(s, p) \in M$ and $(t, p) \in M$ imply that either $s = t$ or $|s-t| \geqslant \varepsilon$. Let A be the set of all $(s, p) \in M$ such that there exists $t > s$ with $(t, p) \in M$. By our assumption, the natural projection $p_N\colon \boldsymbol{R} \times N \to N$ restricted to M is locally diffeomorphic, so $p_N\colon M \to N$ is a covering in view of Corollary 4.7 of [2], p. 178. Therefore, A is open in M. Now let $(s_m, p_m) \in A$, $(s_m, p_m) \to (s, p)$, and choose $t_m > s_m$ such that $(t_m, p_m) \in M$. Let (t, p) be the limit of some convergent subsequence of (t_m, p_m). The inequality $|t_m - s_m| \geqslant \varepsilon$ yields $t - s \geqslant \varepsilon > 0$, so $(s, p) \in A$. Hence A is also closed in M. Suppose that $A = M$. Then for any $p \in N$ we could find a sequence s_m of real numbers such that $s_{m+1} - s_m \geqslant \varepsilon$ and $(s_m, p) \in M$ for any positive integer m, which would contradict the compactness of M. Therefore A is empty, which completes the proof.

Using Theorem 1 and Lemma 1 we obtain immediately

COROLLARY 1. *If the quotient $\bar{M}/f$ is a Hausdorff space and M is a compact spacelike submanifold of $\bar{M}$, then*

(i) *M intersects each maximal integral curve of df at exactly one point;*

(ii) *M is diffeomorphic to $\bar{M}/f$, in particular, any two compact spacelike submanifolds of $\bar{M}$ are diffeomorphic.*

The assertion of Theorem 1 can be strengthened under some additional assumptions about the time function f. An example of this kind is given by the following

PROPOSITION 3. *If df is a complete vector field and $g(df, df) < c$ for some constant $c < 0$, then there exists a diffeomorphism Q of $\bar{M}$ onto $\boldsymbol{R} \times f^{-1}(0)$ such that the diagram*

$$\begin{tikzcd} \bar{M} \arrow[r, "Q"] \arrow[dr, "f"'] & \boldsymbol{R} \times f^{-1}(0) \arrow[d, "p_{\boldsymbol{R}}"] \\ & \boldsymbol{R} \end{tikzcd}$$

commutes, $p_{\boldsymbol{R}}$ being the natural projection, and Q maps the maximal integral curve of df through any $p \in f^{-1}(0)$ onto $\boldsymbol{R} \times \{p\}$. In particular, $\bar{M}/f$ is a Hausdorff space and it is diffeomorphic to each of the submanifolds $f^{-1}(t)$ of $\bar{M}$ for $t \in \boldsymbol{R}$.

Proof. Let ψ_t be the flow of df and consider a maximal integral curve $\boldsymbol{R} \ni t \mapsto \psi_t p$. The inequality

$$\frac{d}{dt} f(\psi_t p) = g(df, df) < c < 0$$

shows that, along this curve, f assumes all real values. Therefore, for any $p \in \bar{M}$ there exists exactly one $t(p) \in \boldsymbol{R}$ such that $f(\psi_{t(p)} p) = 0$. Applying the implicit function theorem to the assignment $(t, p) \mapsto f(\psi_t p)$ we conclude that $p \mapsto t(p)$ is a C^∞-function. Now define $Q\colon \bar{M} \to \boldsymbol{R} \times f^{-1}(0)$ by $Q(p) = \big(f(p), \psi_{t(p)} p\big)$. It is clear that Q is one-one and maps onto $\boldsymbol{R} \times f^{-1}(0)$. For a vector $Y = \dot{p}_0 \in T_p\bar{M}$ we have

$$Q_* Y = \frac{d}{ds}\big(f(p_s), \psi_{t(p_s)} p_s\big)\big|_{s=0} = \big(df(Y), (\psi_{t(p)})_* Y + dt(Y)df\big).$$

Suppose that $Q_* Y = 0$. Then $df(Y) = 0$, i.e., Y is orthogonal to df, and $(\psi_{t(p)})_* Y = -dt(Y)df$ is tangent to an integral curve of df, hence so is $Y = (\psi_{-t(p)})_*(\psi_{t(p)})_* Y$, which yields $Y = 0$. Therefore, Q is a diffeomorphism and it is easy to see that it has all the desired properties, which completes the proof.

Given a timelike 1-form ω on $\bar{M}$, the orthogonal complement $\omega^\perp$ of ω is an n-dimensional spacelike distribution of class C^∞ on $\bar{M}$. If ω is closed, that is $d\omega = 0$, then $\omega^\perp$ is *involutive*, i.e., through each point of $\bar{M}$ there passes an integral manifold of $\omega^\perp$. In fact, such a manifold can be defined by $F = \text{const}$, where F is a function with $dF = \omega$ in a neighbourhood of the given point. For a closed timelike 1-form ω on $\bar{M}$, each point $p \in \bar{M}$ lies in a unique maximal integral manifold of $\omega^\perp$ (see [1], p. 88-95).

PROPOSITION 4. *Suppose that $\bar{M}/f$ is a Hausdorff space. If ω is a closed timelike 1-form on $\bar{M}$ such that some maximal integral manifold N of $\omega^\perp$ is compact, then ω is the gradient of some time function F on $\bar{M}$.*

Proof. Let $p \in \bar{M}$. By (i) of Corollary 1, N intersects the curve $t \mapsto \varphi_t p$ for exactly one parameter value. We define the function $T\colon \bar{M} \to \boldsymbol{R}$

by the relation $\varphi_{T(p)}p \in N$. For any $t \in \boldsymbol{R}$ and $p \in \bar{M}$ we have clearly $T(\varphi_t p) = T(p) - t$. Given $p \in \bar{M}$, let $\bar{F}$ be a primitive function of ω in a neighbourhood of $\varphi_{T(p)}p$ such that $\bar{F}(\varphi_{T(p)}p) = 0$. Applying the implicit function theorem to the assignment $(t, q) \mapsto \bar{F}(\varphi_t q)$, we conclude that the function T is of class C^∞. Now we define the C^∞-function $F\colon \bar{M} \to \boldsymbol{R}$ by

$$F(p) = -\int_0^{T(p)} \omega\big(X(\varphi_t p)\big)dt = -\int_0^{T(p)} \omega\left(\frac{d}{dt}\varphi_t p\right)dt.$$

Suppose that the points p and q lie in an integral manifold P of $\omega^\perp$ and choose a piecewise C^∞-curve $x\colon [0,1] \to P$ with $x(0) = p$ and $x(1) = q$. Set $S = [0,1] \times [0,1]$. The formula $K(s,t) = \varphi_{tT(x(s))}x(s)$ defines a piecewise C^∞-mapping $K\colon S \to \bar{M}$. Using the fact that ω is orthogonal to the curve x as well as to the curve $s \mapsto \varphi_{T(x(s))}x(s)$, and applying the Stokes formula, we obtain

$$\begin{aligned} 0 &= \int_S K^* d\omega = \int_0^1 \omega\left(\frac{d}{dt}\varphi_{tT(q)}q\right)dt - \int_0^1 \omega\left(\frac{d}{dt}\varphi_{tT(p)}p\right)dt \\ &= \int_0^{T(q)} \omega\left(\frac{d}{dt}\varphi_t q\right)dt - \int_0^{T(p)} \omega\left(\frac{d}{dt}\varphi_t p\right)dt = F(p) - F(q). \end{aligned}$$

Therefore, F is constant along any integral manifold of $\omega^\perp$. Thus, for any $p \in \bar{M}$, $dF_p = c\omega_p$ for some real number c, since both vectors dF_p and ω_p are orthogonal to $\omega_p^\perp$.

Consider the vector

$$X_p = \frac{d}{ds}\varphi_s p|_{s=0}.$$

We have

$$\begin{aligned} F(\varphi_s p) &= -\int_0^{T(\varphi_s p)} \omega\big(X(\varphi_t\varphi_s p)\big)dt = -\int_0^{T(p)-s} \omega\big(X(\varphi_{t+s}p)\big)dt \\ &= -\int_s^{T(p)} \omega\big(X(\varphi_t p)\big)dt = \int_{T(p)}^{s} \omega\big(X(\varphi_t p)\big)dt, \end{aligned}$$

hence

$$dF_p(X_p) = \frac{d}{ds}F(\varphi_s p)|_{s=0} = \omega_p(X_p) \neq 0,$$

which shows that $c = 1$ and $dF_p = \omega_p$. Therefore $dF = \omega$, which completes the proof.

3. The non-Hausdorff case. It will be shown in the sequel that both assertions of Corollary 1 fail in general if $\bar{M}/f$ is not a Hausdorff space. Moreover, no additional assumptions about the topology of the compact spacelike submanifolds can help it.

First we prove

LEMMA 2. *Let M be an n-dimensional Hausdorff manifold. Then there exist open subsets U and V of M with disjoint compact closures and a positive definite metric h on M such that both (U, h) and (V, h) are isometric to the open unit ball in the Euclidean space $\boldsymbol{R}^n$.*

Proof. Let h' be any positive definite metric on M. Choose open subsets U' and V' with disjoint compact closures and a positive definite metric h'' on $U' \cup V'$ such that both (U', h'') and (V', h'') are isometric to the n-ball of radius 2. Let $U \subset U'$ and $V \subset V'$ be the subsets corresponding to the unit n-ball in the above isometries. Now let F be a C^∞-function on M such that $0 \leqslant F \leqslant 1$, $F = 0$ on $U \cup V$ and $F = 1$ on an open set containing $M - (U' \cup V')$. It is easy to see that U and V together with the metric $h = Fh' + (1-F)h''$ satisfy our assertion, which completes the proof.

THEOREM 2. *Let M be an n-dimensional compact Hausdorff manifold. Then there exists an $(n+1)$-dimensional Hausdorff manifold $\bar{M}$ with a Riemannian metric g of index one and with a time function f such that M can be embedded in $\bar{M}$ as a spacelike submanifold intersecting some integral curve of df at two points.*

Proof. Choose U, V, and h as in Lemma 2. Let F be a C^∞-function on M such that $F = 0$ on U and $F = 3$ on V. Define the Riemannian metric $\bar{g}$ and the function f on $\boldsymbol{R} \times M$ by

$$(\boldsymbol{R} \times M, \bar{g}) = (\boldsymbol{R}, (dx)^2) \times (M, h) \quad \text{and} \quad f(t, p) = t + F(p).$$

We shall identify M with the submanifold $\{0\} \times M$ of $\boldsymbol{R} \times M$.

Let Df denote the gradient of f in $(\boldsymbol{R} \times M, \bar{g})$. Clearly, $Df \neq 0$ at each point of $\boldsymbol{R} \times M$. For $p \in M$, the vector

$$Y = \frac{d}{dt}(t, p)\Big|_{t=0}$$

is $\bar{g}$-orthogonal to M and satisfies the relation $\bar{g}(Df_p, Y) = 1$. Therefore, Df is transverse to M.

The formula

$$v(Z) = \frac{\bar{g}(Z, Df_p)}{\bar{g}(Df_p, Df_p)} Df_p \quad \text{for } p \in M,\ Z \in T_pM,\ h(Z, Z) = 1$$

defines a differentiable mapping $v\colon T^1M \to T(\boldsymbol{R} \times M)$, T^1M being the space of the unit tangent bundle of (M, h). The Schwarz inequality yields

$$\bar{g}(Z, v(Z)) = \bar{g}(v(Z), v(Z)) < 1,$$

so in view of compactness of T^1M we may choose $c > 0$ such that

$$\bar{g}(Z, v(Z)) < \frac{1}{1+c^2} \quad \text{for any } Z \in T^1M.$$

Now let g be the unique C^∞ Riemannian metric of index one in $\boldsymbol{R} \times M$ such that

$$g(Df, Df) = -c^2\bar{g}(Df, Df), \quad g(Df, Y) = 0 \quad \text{and} \quad g(Y, Y) = \bar{g}(Y, Y)$$

for any vector Y $\bar{g}$-orthogonal to Df. For any $Z \in T^1M$ we have

$$\bar{g}(Df, Z - v(Z)) = 0,$$

whence

$$\begin{aligned} g(Z, Z) &= g(v(Z), v(Z)) + g(Z - v(Z), Z - v(Z)) = -c^2\bar{g}(Z, v(Z)) + \\ &\quad + 1 - \bar{g}(Z, v(Z)) = 1 - (1+c^2)\bar{g}(Z, v(Z)) > 1 - \frac{1+c^2}{1+c^2} = 0. \end{aligned}$$

Thus M is spacelike in $(\boldsymbol{R} \times M, g)$. Let df be the gradient of f in the sense of g. Clearly, $g(df, Y) = 0$ whenever $\bar{g}(Df, Y) = 0$, which shows that df is parallel to Df at each point. In particular, both gradients have the same integral curves up to a change of parameter. Thus f is a time function in $(\boldsymbol{R} \times M, g)$.

Define the open submanifold $\bar{M}^\vee$ of $\boldsymbol{R} \times M$ by

$$\bar{M}^\vee = (-1, 1) \times M \cup (-1, 2) \times U \cup (-2, 1) \times V.$$

The mapping $H: (1, 2) \times U \to (-2, -1) \times V$, given by $H(t, p) = (t-3, H'(p))$, where H' is any isometry of U onto V, is an isometry in the sense of both $\bar{g}$ and g (since F is constant on both U and V) and satisfies the condition $f \circ H = f$. Let $\bar{M}$ be the space obtained from $\bar{M}^\vee$ by identifying $(1, 2) \times U$ with $(-2, -1) \times V$ by means of H. It is easy to see that $\bar{M}$ is an $(n+1)$-dimensional Hausdorff manifold with a C^∞ differentiable structure induced in a natural manner from $\bar{M}^\vee$. Moreover, g and f induce a Riemannian metric of index one and a time function on $\bar{M}$, denoted for simplicity also by g and f. Clearly, M is embedded in $\bar{M}$ in an obvious manner as a spacelike submanifold.

Since F is constant on both U and V, the field df is tangent to the lines $\boldsymbol{R} \times \{p\}$, $p \in U \cup V$. It is now clear that for any $p \in U$ the segments $(-1, 2) \times \{p\}$ and $(-2, 1) \times \{H'(p)\}$ define together an unparametrized integral curve of df in $\bar{M}$ which intersects M at two points, namely at $(0, p)$ and at $(0, H'(p))$. This completes the proof.

For $n = 1$ and $M = S^1$ the above construction is illustrated in Fig. 1. The 2-dimensional manifold $\bar{M}$ is embedded in $\boldsymbol{R}^3$ and its metric g of index one is the one induced from $(dx)^2+(dy)^2-(dz)^2$ by the embedding. Our time function f coincides with the coordinate projection z and the integral curves of df are represented by vertical segments. The spacelike circle in $\bar{M}$ is denoted by the dark line.

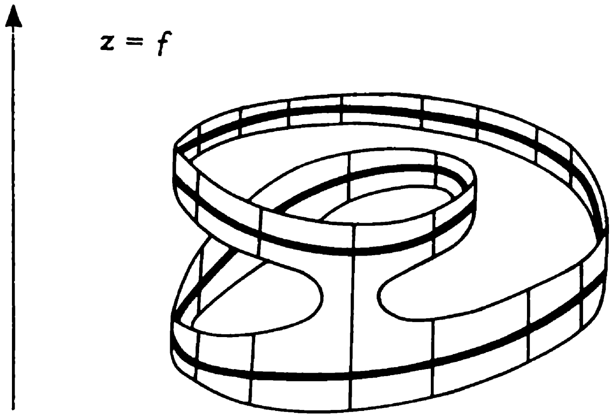

Fig. 1

THEOREM 3. *There exists an $(n+1)$-dimensional Hausdorff manifold $\bar{M}$ with a Riemannian metric g of index one and with a time function f such that each compact n-dimensional Hausdorff manifold can be embedded in $\bar{M}$ as a spacelike submanifold.*

Proof. Since there exist only countably many non-diffeomorphic compact n-dimensional Hausdorff manifolds, we may choose a sequence M_k, $k=1,2,\ldots$, which contains a diffeomorphic image of each such manifold. For each M_k find U_k, V_k, and h_k as in Lemma 2. Let H_k' denote an isometry of V_k onto U_{k+1} and define the positive definite metric $\bar{g}_k$ and the metric of index one g_k on $\boldsymbol{R}\times M_k$ by

$$(\boldsymbol{R}\times M_k,\ \bar{g}_k)=\big(\boldsymbol{R},\ (dx)^2\big)\times(M_k,\ h_k)$$

and

$$(\boldsymbol{R}\times M_k,\ g_k)\ =\big(\boldsymbol{R},\ -(dx)^2\big)\times(M_k,\ h_k).$$

We identify M_k with the g_k-spacelike submanifold $\{k\}\times M_k$ of $\boldsymbol{R}\times M_k$. Clearly, the natural projection

$$f_k=p_{\boldsymbol{R}}\colon\ \boldsymbol{R}\times M_k\to\boldsymbol{R}$$

is a time function in the sense of g_k. Define the open submanifold $\bar{M}_k$ of $\boldsymbol{R}\times M_k$ by

$$\bar{M}_k\ =(k-\tfrac{1}{2},k+\tfrac{1}{2})\times M_k\cup(k-\tfrac{1}{2},k+1)\times U_k\cup(k-1,k+\tfrac{1}{2})\times V_k.$$

The mapping

$$H_k\colon (k, k+1) \times U_k \to (k, k+1) \times V_{k+1},$$

given by $H_k(t, p) = \big(t, H_k'(p)\big)$ is an isometry in the sense of $\bar{g}_k$ and $\bar{g}_{k+1}$ as well as in the sense of g_k and g_{k+1}. Moreover, $f_{k+1} \circ H_k = f_k$. Let $\bar{M}$ be the space obtained from the disjoint union $\bigcup\limits_k \bar{M}_k$ by identifying $(k, k+1) \times U_k \subset \bar{M}_k$ with $(k, k+1) \times V_{k+1} \subset \bar{M}_{k+1}$ by means of H_k, $k = 1, 2, \ldots$ It is easy to see that $\bar{M}$ is an $(n+1)$-dimensional Hausdorff topological manifold and it is provided with a natural C^∞ differentiable structure. Clearly, the metrics g_k and functions f_k induce a C^∞ Riemannian metric g of index one in $\bar{M}$ and a time function f on $(\bar{M}, g)$. For each k, M_k is embedded in $\bar{M}$ as a spacelike submanifold, which completes the proof.

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