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CONFLUENT AND RELATED MAPPINGS

BY

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A *compactum* is a compact Hausdorff space. All spaces considered in this paper * are assumed to be compacta. A *continuum* is a connected compactum. A *mapping* $f\colon X \to Y$ is a continuous function from X to Y. A mapping $f\colon X \to Y$ of X onto Y is said to be *confluent* (see [1], p. 213) if, for each subcontinuum $K \subset Y$ and a component C of $f^{-1}(K)$, $f(C) = K$; and it is said to be *weakly confluent* (see [3]) if, for each subcontinuum $K \subset Y$, there exists a component C of $f^{-1}(K)$ such that $f(C) = K$. Every open mapping is confluent (see [8], p. 148) as is every monotone mapping.

In this paper conditions under which certain mappings are confluent or quasi-interior are investigated, and it is shown that every mapping from a metric continuum onto an arc-like continuum is weakly confluent.

THEOREM 1. *If $f\colon X \to Y$ is a confluent mapping of X onto a continuum Y, then there is a subcontinuum $L \subset X$ such that $f|L$ is a confluent mapping of L onto Y and L is minimal with respect to this property.*

Proof. Let X' be a component of $f^{-1}(Y)$ and

$$\mathscr{C} = \{H\colon H \text{ is a subcontinuum of } X' \text{ and } f|H \text{ is a confluent mapping of } H \text{ onto } Y\}.$$

Let $\mathscr{D}$ be a maximal totally ordered (by inclusion) subcollection of $\mathscr{C}$ and

$$L = \bigcap_{H \in \mathscr{D}} H.$$

Then, L is clearly a continuum which is mapped onto Y by f. Let K be a subcontinuum of Y and let C be a component in L of $f^{-1}(K)$. For each $H \in \mathscr{D}$, let C_H be the component in H of $f^{-1}(K)$ which contains C. Thus $f(C_H) = K$ for each $H \in \mathscr{D}$. Clearly,

$$C = \bigcap_{H \in \mathscr{D}} C_H$$

* Almost all the results of this paper are contained in author's doctoral dissertation [7] which was partially supported by a National Science Foundation Science Faculty Fellowship.

so $f(C) = K$. Thus $f|L$ is a confluent mapping of L onto Y and, by construction, L is minimal with respect to this property.

A mapping $f: X \to Y$ of a metric compactum X onto a metric compactum Y such that if $y \in Y$, C is a component of $f^{-1}(y)$, and U is an open set containing C, then $y \in \operatorname{Int}(f(U))$, is called *quasi-interior* [9]. Lelek and Read [4] showed that every quasi-interior mapping is confluent.

THEOREM 2. *Let X, Y, W and Z be metric compacta. If $f: X \to W$ is a quasi-interior mapping of X onto W and $g: Y \to Z$ is a quasi-interior mapping of Y onto Z, then*

$$f \times g: X \times Y \to W \times Z$$

is a quasi-interior mapping of $X \times Y$ onto $W \times Z$.

Proof. Since f and g are quasi-interior, $f = rs$ and $g = tu$ with r and t light and open and s and u monotone [9]. Thus

$$f \times g = (r \times t)(s \times u)$$

with $r \times t$ light and open and $s \times u$ monotone. Hence $f \times g$ is quasi-interior [ibidem].

A similar result holds for confluent mappings onto locally connected metric spaces:

COROLLARY. *Let X and Y be metric compacta and let W and Z be locally connected metric spaces. If $f: X \to W$ is a confluent mapping of X onto W and $g: Y \to Z$ is a confluent mapping of Y onto Z, then*

$$f \times g: X \times Y \to W \times Z$$

is a confluent mapping of $X \times Y$ onto $W \times Z$.

Proof. By a result of Lelek and Read [4], confluent mappings onto locally connected spaces are quasi-interior. Hence, by Theorem 2, $f \times g$ is quasi-interior and thus confluent.

A continuum is *hereditarily unicoherent* if the common part of each two of its subcontinua is connected. It is known [4] that if $f: X \to Y$ is a mapping of the continuum X onto the hereditarily unicoherent continuum $Y = Y_1 \cup \ldots \cup Y_n$, where Y_i are continua such that the mappings $f|f^{-1}(Y_i)$ are confluent ($i = 1, \ldots, n$), then f is confluent. An example of Lelek (see [4], Example 4.2) can be used to show that f need not be confluent if

$$Y = \bigcup_{n=0}^{\infty} Y_n$$

with the mappings $f|f^{-1}(Y_n)$ confluent ($n = 0, 1, \ldots$).

Example. Put

$$X = \{(x, 2):\ -1 \leqslant x \leqslant 2\} \cup \left\{\left(\sin\frac{1}{y-2}, y\right):\ 2 < y \leqslant 3\right\} \cup$$
$$\cup \{(2, y):\ -1 \leqslant y \leqslant 2\} \cup \left\{\left(x, \sin\frac{1}{x-2}\right):\ 2 < x \leqslant 3\right\},$$

and let R be the equivalence relation defined by

$$R = \{((t, 2), (2, t)):\ -1 \leqslant t \leqslant 2\} \cup$$
$$\cup \{((2, t), (t, 2)):\ -1 \leqslant t \leqslant 2\} \cup \{(p, p):\ p \in X\}.$$

Let f be the projection mapping of X onto $Y = X/R$. Let $A_0 = \{(x, 2):\ -1 \leqslant x \leqslant 2\}$ and, for each positive integer n, let A_{2n} be the intersection of X with the strip of the plane determined by the inequality

$$2+\frac{1}{n+1} \leqslant x \leqslant 2+\frac{1}{n},$$

and let A_{2n-1} be the intersection of X with the strip determined by

$$2+\frac{1}{n+1} \leqslant y \leqslant 2+\frac{1}{n}.$$

Let $Y_n = f(A_n)$ $(n = 0, 1, \ldots)$. Clearly,

$$Y = \bigcup_{n=0}^{\infty} Y_n$$

is hereditarily unicoherent, each Y_n is a continuum, and the mappings $f|f^{-1}(Y_n)$ are confluent. Let X_1 and X_2 be the intersections of X with the strips determined by the inequalities $|x| \leqslant 1$ and $|y| \leqslant 1$, respectively. No subcontinuum of X maps onto $K = f(X_1 \cup X_2)$. Hence, f is not confluent.

A similar result to the finite case holds for quasi-interior mappings.

THEOREM 3. *Let f: $X \to Y = Y_1 \cup Y_2$ be a mapping from the metric compactum X onto the metric space Y, with Y_1 and Y_2 compact. If $f|f^{-1}(Y_1)$ and $f|f^{-1}(Y_2)$ are quasi-interior, then f is quasi-interior.*

Proof. Let $y \in Y$, C be a component of $f^{-1}(y)$, and let U be an open set containing C. Suppose, by the way of contradiction, that $y \notin \operatorname{Int}(f(U))$. Then y is the limit of a sequence y_n of elements of $Y \setminus f(U)$. If there is a subsequence y_{n_i} of y_n such that, for each i, $y_{n_i} \in Y_1$, then $y \in Y_1$. Thus, $U \cap f^{-1}(Y_1)$ is open in $f^{-1}(Y_1)$ and contains C. Hence,

$$y \in \operatorname{Int}[f(U \cap f^{-1}(Y_1))]$$

relative to Y_1. Thus, there is an n_j such that

$$y_{n_j} \in \operatorname{Int}[f(U \cap f^{-1}(Y_1))]$$

relative to Y_1 which is the desired contradiction. A similar contradiction is obtained if there is a subsequence y_{n_i} such that, for each i, $y_{n_i} \in Y_2$. Hence $y \in \mathrm{Int}(f(U))$, and thus f is quasi-interior.

A metric continuum X is said to be *arc-like* (respectively, *tree-like*) if, for each positive number b, there is an arc (respectively, tree) I and a mapping $f\colon X \to I$ from X onto I such that if $z \in I$, then $\mathrm{diam}(f^{-1}(z)) \leqslant b$. If

$$X = \prod_{i \in \Delta} X_i$$

is a product space, then p_i denotes the projection mapping from X onto X_i. The following lemma together with a characterization by Mardešić and Segal [5] can be used to show that every mapping from a metric continuum onto an arc-like continuum is weakly confluent:

LEMMA. *If $f\colon X \to I = [0,1]$ is a mapping from the continuum X onto I, then f is weakly confluent.*

Proof. Let $[a, b]$ be a subcontinuum of I. Let $K = \{(x, f(x))\colon x \in X\}$. Clearly, K is a subcontinuum of $X \times I$. Suppose, by the way of contradiction, that $K \cap (X \times [a, b])$ does not contain a continuum irreducible between $K \cap (X \times \{a\})$ and $K \cap (X \times \{b\})$. Then

$$K \cap (X \times [a, b]) = P \cup Q,$$

with P and Q disjoint closed sets containing $K \cap (X \times \{a\})$ and $K \cap (X \times \{b\})$, respectively (see [6], p. 15). Let

$$P' = P \cup [p_2^{-1}([0, a]) \cap K] \quad \text{and} \quad Q' = Q \cup [p_2^{-1}([b, 1]) \cap K].$$

Then $K = P' \cup Q'$, which is a contradiction, since P' and Q' are mutually separated. Hence, there is a subcontinuum L of $K \cap (X \times [a, b])$ irreducible between $K \cap (X \times \{a\})$ and $K \cap (X \times \{b\})$. Clearly, $f(p_1(L)) = [a, b]$. Thus f is weakly confluent.

THEOREM 4. *If $f\colon X \to Y$ is a mapping of the metric continuum X onto an arc-like continuum Y, then f is weakly confluent.*

Proof. Let $f\colon X \to Y$ be a mapping of X onto Y. By a theorem of Mardešić and Segal [5], there exist a sequence Y_i of arcs and, for $i \leqslant j$, a mapping p_i^j from Y_j onto Y_i such that

(1) p_i^i is the identity mapping of Y_i onto Y_i,

(2) if $i \leqslant j \leqslant k$, then $p_i^k = p_i^j p_j^k$, and

(3) $Y = \varprojlim (Y_i, p_i^j) = \{z\colon z \in \prod_{i=1}^{\infty} Y_i$ and, if $i \leqslant j$, $p_i^j(p_j(z)) = p_i(z)\}$.

Since each Y_i is homeomorphic to $[0, 1]$, it can be assumed, without loss of generality, that $Y_i = [0, 1]$ $(i = 1, \ldots)$. Let C be a subcontinuum of Y. By the lemma, $p_i f$ is weakly confluent for each i. Thus, since $C_i = p_i(C)$ is a subcontinuum of Y_i, there exists a subcontinuum $K_i \subset X$ such that $p_i f(K_i) = C_i$ for each i. There exists a subsequence K_{n_i} which converges to a limit subcontinuum K of X. Since Y is homeomorphic to $\varprojlim(Y_{n_i}, p_{n_i}^{n_j})$, suppose, without loss of generality, that K is the limit of the sequence K_i.

Let $y \in C$. For each i, let $x_i \in K_i$ such that $p_i f(x_i) = p_i(y)$. Then there is an $x \in K$ such that x is a cluster point of the sequence x_i. Suppose, by the way of contradiction, that $f(x) \neq y$. Then, there is an n such that $p_n f(x) \neq p_n(y)$. Hence, there exist disjoint open sets U and V such that $p_n f(x) \in U$ and $p_n(y) \in V$. Thus $f^{-1}[p_n^{-1}(U)]$ is an open subset of X containing x. Hence, there exists an $m > n$ such that $x_m \in f^{-1}[p_n^{-1}(U)]$. Therefore,

$$p_n(y) = p_n^m p_m(y) = p_n^m p_m f(x_m) = p_n[f(x_m)] \in U,$$

which is the desired contradiction. Hence, $f(x) = y$ and thus $C \subset f(K)$.

Let $x \in K$. There exists a sequence x_i having the limit x such that each $x_i \in K_i$. Thus $p_i f(x_i) \in C_i$ $(i = 1, \ldots)$. For each i, let

$$z_i \in p_i^{-1}[p_i f(x_i)] \cap C.$$

Since C is compact, there is a $z \in C$ such that z is a cluster point of the sequence z_i. By an argument similar to that above-given, the assumption that $f(x) \neq z$ leads to a contradiction. Hence $f(K) = C$, and thus f is weakly confluent.

A mapping f of a space X onto a space Y is said to be *locally confluent* [2] if, for each point $y \in Y$, there is an open set $O \subset Y$ containing y such that $f|f^{-1}(\bar{O})$ is confluent. It has been shown [4] that all locally confluent mappings onto hereditarily arc-wise connected spaces are confluent, and further, if $f\colon X \to Y$ is locally confluent and $B \subset Y$ is a closed subset, then

$$f|f^{-1}(B)\colon f^{-1}(B) \to B$$

is locally confluent. The arc-wise connected component of a point p of a space X is the union of all paths (i.e. continuous images of $[0,1]$) in X which contain p. The following theorem answers a question raised by A. Lelek in a letter to the author:

THEOREM 5. *If $f\colon X \to Y$ is a locally confluent mapping of the metric continuum X onto the tree-like continuum Y, with Y having no more than two arc-wise connected components, then f is weakly confluent.*

Proof. Clearly, Y can contain no indecomposable continuum. If Y has only one arc-wise connected component, then Y is hereditarily arc-wise connected (since Y is hereditarily unicoherent), and thus f is confluent. Hence, suppose that Y has exactly two arc-wise connected components, A and B. Let K be a subcontinuum of Y. If $K \subset A$ or $K \subset B$, then K is hereditarily arc-wise connected, $f|f^{-1}(K)$ is confluent, and each component of $f^{-1}(K)$ maps onto K. Thus, suppose that $p, q \in K$ with $p \in A \setminus B$ and $q \in B \setminus A$. Since

$$K = (A \cap K) \cup (B \cap K),$$

there exists a $z \in \overline{(A \cap K)} \cap (B \cap K)$ or a $z \in (A \cap K) \cap \overline{(B \cap K)}$. Suppose, without loss of generality, that $z \in (A \cap K) \cap \overline{(B \cap K)}$. Let C be a component of $f^{-1}(K)$ such that $C \cap f^{-1}(q) \neq \emptyset$. Let $w \in C \cap f^{-1}(q)$. There is a sequence z_i of points of $B \cap K$ such that z_i has the limit z. For each i, let K_i be the (unique) arc between q and z_i. Since Y is hereditarily unicoherent, each $K_i \subset B \cap K$. Let H_i be the component of $f^{-1}(K_i)$ $(i = 1, \ldots)$ containing w. Each K_i is hereditarily arc-wise connected, so $f|f^{-1}(K_i)$ is confluent. Thus $f(H_i) = K_i$ $(i = 1, \ldots)$. Let

$$H = \overline{\bigcup_{i=1}^{\infty} H_i}.$$

Clearly, H is a subcontinuum of $f^{-1}(K)$, and thus of C. For each positive integer i, let $x_i \in H_i \cap f^{-1}(z_i)$. There exists a cluster point $x \in H$ of the sequence x_i, and, clearly, $f(x) = z$. Let $y \in K$. If $y \in A$, let $K' \subset A \cap K$ be the arc between z and y. Then $f|f^{-1}(K')$ is confluent. Thus, if C' is the component of $f^{-1}(K')$ containing x, then $C' \subset C$ and $f(C') = K'$, so $y \in f(C)$. If $y \in B$, let K'' be the arc between q and y. Then $K'' \subset B \cap K$ and $f|f^{-1}(K'')$ is confluent. Hence, if C'' is the component of $f^{-1}(K'')$ containing w, then $C'' \subset C$ and $f(C'') = K''$. Hence $y \in f(C)$. Thus $f(C) = K$ and f is weakly confluent.

An analogue of Theorem 5 is not true for Y having three arc-wise connected components (see [4], Example 4.2).

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