COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

AN INTEGRAL REPRESENTATION OF LIMIT LAWS

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1. Notation and preliminaries. Let $X_1, X_2, \ldots$ be a sequence of real-valued, independent and identically distributed random variables. It was proved by P. Lévy [7] that all possible limit laws of suitably normed sums

$$a_n(X_1 + X_2 + \ldots + X_n) + b_n \tag{1.1}$$

with $a_n \in (0, \infty)$ and $b_n \in (-\infty, \infty)$ form the family of stable laws. A. Ya. Khintchine [3] showed that every infinitely divisible law can be obtained as the limit of a subsequence of probability distributions of (1.1). W. Feller [2] restricted the summands to be in a class which makes the normed sums (1.1) stochastically compact, i.e. so that the sequence of probability distributions of (1.1) is conditionally compact and all its cluster points are nondegenerate laws. Let $\mathbf{F}$ be the family of all possible cluster points for sequences obeying Feller's condition. It is clear that $\mathbf{F}$ contains all stable laws and is contained in the family of infinitely divisible laws. In other words, the characteristic function of a probability measure P from $\mathbf{F}$ is given by the Lévy–Khintchine formula

$$\hat{P}(t) = \exp\left(iat + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \frac{N(dx)}{m(x)}\right), \tag{1.2}$$

where $a \in (-\infty, \infty)$, $m(x) = \min(x^2, 1)$ and N is a finite Borel measure on the real line. It is evident that the relation $P \in \mathbf{F}$ does not depend upon the parameter a. Denote by $\mathbf{H}$ the family of all measures N corresponding in (1.2) to probability laws from $\mathbf{F}$. A nice analytic characterization of the family $\mathbf{H}$ has been given by W. E. Pruitt [8].

PRUITT'S THEOREM. *A measure N belongs to $\mathbf{H}$ if and only if it does not vanish identically and there exists a positive number c such that*

$$x^2 \int_{|y|>x} \frac{N(dy)}{m(y)} \le c \int_{|y|\le x} \frac{y^2}{m(y)} N(dy) \tag{1.3}$$

for all $x \in (0, \infty)$.

We shall lean heavily on this theorem, which provides a tool for investigating the family $\mathbf{H}$.

Let $\mathbf{F}_{sym}$ be the subset of $\mathbf{F}$ consisting of symmetric probability distributions. It is clear that these distributions can be obtained as cluster points for the sequences of probability distributions of (1.1) with $b_n = 0$ and symmetrically distributed X_n $(n = 1, 2, \ldots)$. Of course, each measure from $\mathbf{F}_{sym}$ is of the form $e(M)$ where

$$\hat{e}(M)(t) = \exp \int_0^\infty (\cos tx - 1) \frac{M(dx)}{m(x)} \tag{1.4}$$

and M is a finite Borel measure on the half-line $[0, \infty)$. Put $\mathbf{H}_{sym} = \{ M : e(M) \in \mathbf{F}_{sym} \}$. By Φ we denote the set of all Borel mappings from $(0, \infty)$ into $[0, 1]$ and by δ_a the probability measure concentrated at the point a. The symbol $-A$ is used for the set $\{ -x : x \in A \}$. The following simple statement reduces the study of limit laws to the symmetric case.

PROPOSITION 1.1. *A finite Borel measure N belongs to $\mathbf{H}$ if and only if there exist a measure M in $\mathbf{H}_{sym}$ and a function φ in Φ such that*

$$\begin{aligned} N(A) = M(\{0\})\delta_0(A) &+ \int_{A \cap (0,\infty)} \varphi(x)\, M(dx) \\ &+ \int_{(-A)\cap(0,\infty)} (1 - \varphi(x))\, M(dx) \end{aligned} \tag{1.5}$$

for all Borel subsets A of $(-\infty, \infty)$.

Proof. Given $M \in \mathbf{H}_{sym}$ and $\varphi \in \Phi$ we define the measure N by means of (1.5). Then we have the formulae

$$\int_{|y|>x} \frac{N(dy)}{m(y)} = \int_{x+}^{\infty} \frac{M(dy)}{m(y)}, \qquad \int_{|y|\le x} \frac{y^2}{m(y)} N(dy) = \int_0^x \frac{y^2}{m(y)} M(dy) \tag{1.6}$$

for $x \in (0, \infty)$, which, by Pruitt's Theorem, yields $N \in \mathbf{H}$.

Suppose now that $N \in \mathbf{H}$ and put for any Borel subset B of $[0, \infty)$

$$M(B) = N(\{0\})\delta_0(B) + N(B \cap (0, \infty)) + N((-B) \cap (0, \infty)). \tag{1.7}$$

It is easy to check (1.6), which, by Pruitt's Theorem, shows that $M \in \mathbf{H}_{sym}$. Since $N(B \cap (0, \infty)) \le M(B)$ for Borel subsets B of $[0, \infty)$, we have, by the Radon–Nikodym Theorem, the existence of a density function $\varphi \in \Phi$ such that $N(B \cap (0, \infty)) = \int_{B\cap(0,\infty)} \varphi(x)\, M(dx)$. By (1.7) we get $N(\{0\}) = M(\{0\})$ and $N((-B) \cap (0, \infty)) = \int_{(-B)\cap(0,\infty)} (1 - \varphi(x))\, M(dx)$ for Borel subsets B of $[0, \infty)$. From this (1.5) follows, which completes the proof.

The aim of this paper is to give an integral representation for measures belonging to $\mathbf{H}_{\mathrm{sym}}$. The method of proof consists in finding the extreme points of a certain convex compact set formed by measures from $\mathbf{H}_{\mathrm{sym}}$. Once the extreme points are found one can apply a theorem by Choquet on representation of the points of a compact convex set as barycenters of the extreme points.

Before proceeding to state and prove the main results of this paper we shall establish auxiliary propositions.

2. Some functions associated with measures. Given a finite Borel measure M on $[0,\infty)$ and $x \in (0,\infty)$ we put

$$i(M,x) = \int_{x}^{\infty} \frac{M(dy)}{m(y)}, \qquad j(M,x) = \int_{0}^{x-} \frac{y^2}{m(y)} M(dy), \tag{2.1}$$

$$k(M,x) = \frac{1}{2} i(M,x) + \frac{1}{2} x^2 j(M,x). \tag{2.2}$$

It is easy to check the formulae

$$i(M,x) = \int_{x}^{\infty} y^{-2}\, dj(M,y), \tag{2.3}$$

$$\lim_{x\to\infty} x^{-2} j(M,x) = 0, \tag{2.4}$$

$$k(M,x) = \int_{x}^{\infty} y^{-3} j(M,y)\, dy. \tag{2.5}$$

Hence the function $k(M,\cdot)$ is differentiable on $(0,\infty)$ and fulfils the equations

$$i(M,x) = xk'(M,x) + 2k(M,x), \tag{2.6}$$

$$j(M,x) = -x^3 k'(M,x). \tag{2.7}$$

In what follows $S(M)$ will denote the support of the measure M, i.e. the smallest closed subset B of $[0,\infty)$ with $M([0,\infty)\setminus B) = 0$. Further, $A(M)$ will denote the set of atoms of M. It is evident that the functions $i(M,\cdot)$ and $j(M,\cdot)$ are continuous from the left and have right-hand limits on $(0,\infty)$. Moreover, the sets of their discontinuity points coincide with $A(M) \cap (0,\infty)$.

Denote by $\mathbf{G}$ the set of all measures M with $0 \in S(M)$. Since $j(M,x) > 0$ for all $M \in \mathbf{G}$ and $x \in (0,\infty)$, the function

$$l(M,x) = \frac{x^2 i(M,x)}{j(M,x)} \tag{2.8}$$

is well defined, continuous from the left and has a right-hand limit $l(M,x+)$

at every $x \in (0,\infty)$. A simple calculation shows that

$$l(M,x+) \le l(M,x) \tag{2.9}$$

for all $x \in (0,\infty)$ and strict inequality holds if and only if $x \in A(M) \cap (0,\infty)$.

Given a closed subset A of $[0,\infty)$ we put $w(A) = \sup A$. In what follows the symbols $[0, w(A)]$ and $(0, w(A)]$ for $w(A) = \infty$ will denote $[0,\infty)$ and $(0,\infty)$ respectively. For any $x \in [0, w(A)]$ the function $n(A,x) = \min A \cap [x,\infty)$ is well defined. Obviously, it is continuous from the left and has a right-hand limit at every $x \in [0, w(A))$. Moreover, $n(A,x) = x$ if and only if $x \in A$. Also, $n(A,x) \ge x$ for all $x \in [0, w(A)]$.

Suppose that $M \in \mathbf{G}$ and $A \supset S(M)$. Then $M([x, n(A,x))) = 0$ for $x \in [0, w(A)]$, which, by the definition (2.8), yields

$$l(M,x) = \frac{x^2}{n^2(A,x)}\, l(M, n(A,x)) \quad \text{for } x \in (0, w(S(M))]. \tag{2.10}$$

We are now in a position to establish the basic property of the function l.

PROPOSITION 2.1. *Suppose that $M_1, M_2 \in \mathbf{G}$ and*

$$l(M_1,x) = l(M_2,x) \tag{2.11}$$

for $x \in S(M_1) \cup S(M_2)$. Then M_1 is proportional to M_2.

Proof. It is evident, by (2.8), that $l(M_j,x) = 0$ for $x \in (w(S(M_j)),\infty)$ $(j = 1,2)$, which, by (2.11), yields $w(S(M_1)) = w(S(M_2))$. Setting $A = S(M_1) \cup S(M_2)$ and using (2.10) we conclude that (2.11) is fulfilled for all $x \in (0,\infty)$. By (2.6)–(2.8) we have

$$l(M_j,x) = -1 - \frac{2k(M_j,x)}{x k'(M_j,x)}$$

for $j = 1,2$ and $x \in (0,\infty)$. Consequently, (2.11) can be written in the form

$$\frac{k'(M_1,x)}{k(M_1,x)} = \frac{k'(M_2,x)}{k(M_2,x)}$$

for $x \in (0,\infty)$. This proves that $k(M_1,\cdot)$ is proportional to $k(M_2,\cdot)$, which yields the desired assertion.

Suppose that $M \in \mathbf{H}_{\mathrm{sym}}$. Since M does not vanish identically, we have, by (2.1), $i(M,x) > 0$ for sufficiently small x, which, by Pruitt's Theorem, yields $j(M,x) > 0$. Thus $M([0,x)) > 0$ for sufficiently small x and, consequently, $0 \in S(M)$. This proves the inclusion

$$\mathbf{H}_{\mathrm{sym}} \subset \mathbf{G}.$$

Given $c > 0$ we denote by $\mathbf{G}_c$ the subset of $\mathbf{G}$ consisting of measures M fulfilling the condition

$$\sup\{\, l(M,x) : x \in (0,\infty)\,\} \le c.$$

It is clear that in the symmetric case Pruitt's Theorem can be rewritten in the form

$$\mathbf{H}_{\mathrm{sym}} = \bigcup_{c>0} \mathbf{G}_c, \tag{2.12}$$

which reduces the study of $\mathbf{H}_{\mathrm{sym}}$ to that of $\mathbf{G}_c$ for all $c > 0$.

Since $l(aM, x) = l(M, x)$ for $a > 0$ and

$$l(M_1 + M_2, x) = \frac{j(M_1, x)}{j(M_1, x) + j(M_2, x)} l(M_1, x) + \frac{j(M_2, x)}{j(M_1, x) + j(M_2, x)} l(M_2, x) \tag{2.13}$$

the set $\mathbf{G}_c$ is closed under addition and multiplication by positive numbers. Suppose that $M_n \in \mathbf{G}_c$ and $M_n \to M$. Then $l(M_n, x) \to l(M, x)$ for $x \in (0, \infty) \setminus A(M)$. Hence $l(M, x) \le c$ for $x \in (0, \infty) \setminus A(M)$, which, by the continuity from the left of $f(M, \cdot)$, yields $M \in \mathbf{G}_c$. Thus $\mathbf{G}_c$ is closed under passages to the limit.

Given $M \in \mathbf{G}$ and $c > 0$ we put

$$T(M, c) = \{0\} \cup \{\, x : x > 0,\ l(M, x) = c\,\}.$$

Taking into account the continuity from the left of $l(M, \cdot)$ and formula (2.9) we infer that $T(M, c)$ is closed for $M \in \mathbf{G}_c$. Setting $A = S(M)$ and applying (2.10) we get $l(M, x) < c$ if $M \in \mathbf{G}_c$ and $x \notin S(M)$. Consequently, for any $M \in \mathbf{G}_c$

$$T(M, c) \subset S(M). \tag{2.14}$$

Given a sequence A_n of subsets of $[0, \infty)$ we denote by $\mathrm{Li}_{n\to\infty} A_n$ and $\mathrm{Ls}_{n\to\infty} A_n$ the lower and upper topological limit respectively ([5], Section 29). If $\mathrm{Li}_{n\to\infty} A_n = \mathrm{Ls}_{n\to\infty} A_n = A$, then we say that the topological limit $\mathrm{Lim}_{n\to\infty} A_n = A$ exists.

Let f be a function which assigns closed subsets of $[0, \infty)$ to measures from $\mathbf{G}$. The function f is said to be *lower* (respectively *upper*) *semicontinuous* if $M_n \to M$ yields $\mathrm{Li}_{n\to\infty} f(M_n) \supset f(M)$ (respectively $\mathrm{Ls}_{n\to\infty} f(M_n) \subset f(M)$) ([6], Section 43,II). One can easily check that the function $M \to S(M)$ is lower semicontinuous.

LEMMA 2.1. *The function* $\mathbf{G}_c \ni M \to T(M, c)$ *is upper semicontinuous.*

Proof. Suppose that $M_n \in \mathbf{G}_c$ and $M_n \to M$. The limit measure M belongs to $\mathbf{G}_c$ because $\mathbf{G}_c$ is closed. Given an arbitrary subsequence $n_1 < n_2 < \ldots$ and a sequence $x_k \in T(M_{n_k}, c)$ such that $x_k \to x$. It is sufficient to show that $x \in T(M, c)$. Since $0 \in T(M, c)$, we may assume without loss of generality that $x > 0$ and $x_k > 0$ $(k = 1, 2, \ldots)$. For any number $y \notin A(M)$ with $0 < y < x$ we have, by (2.8), $l(M_{n_k}, y) \ge y^2 x_k^{-2} l(M_{n_k}, x_k) = c y^2 x_k^{-2}$

($k = 1,2,\ldots$) and $l(M_{n_k}, y) \to l(M,y)$, which yields $cy^2x^{-2} \leq l(M,y) \leq c$. Letting $y \to x$ and taking into account the continuity from the left of $l(M,\cdot)$ we conclude that $l(M,x) = c$, which completes the proof.

LEMMA 2.2. *Suppose that $M \in \mathbf{G}_c$, $u \in T(M,c)$ and $(u,v) \cap S(M) = \emptyset$ for some $v > u$. Then $u \in A(M)$.*

Proof. In the case $u = 0$ the origin is an isolated point of $S(M)$ and, consequently, belongs to $A(M)$. Suppose now that $u > 0$ and $u \notin A(M)$. Then, by (2.1), $i(M,u) = i(M,v)$ and $j(M,u) = j(M,v)$, which yields

$$c = l(M,u) = u^2v^{-2}l(M,v) \leq cu^2v^{-2}.$$

This contradiction shows that $u \in A(M)$, which completes the proof.

In what follows $\mathbf{Q}_c$ will denote the subset of $\mathbf{G}_c$ consisting of all measures M fulfilling the equation $T(M,c) = S(M)$. Of course, the set $\mathbf{Q}_c$ is closed under multiplication by positive numbers. The lower semicontinuity of $M \to S(M)$ and the upper semicontinuity of $M \to T(M,c)$ on $\mathbf{G}_c$ yield, by (2.14), the following simple statement.

LEMMA 2.3. *The set $\mathbf{Q}_c$ is closed.*

Furthermore, as an immediate consequence of Proposition 2.1 we get

PROPOSITION 2.2. *If $M_1, M_2 \in \mathbf{Q}_c$ and $S(M_1) = S(M_2)$, then M_1 is proportional to M_2.*

3. Extreme points. By $\mathbf{G}_c^1$ and $\mathbf{Q}_c^1$ we shall denote the subsets of $\mathbf{G}_c$ and $\mathbf{Q}_c$ respectively consisting of probability measures.

PROPOSITION 3.1. *The set $\mathbf{G}_c^1$ is convex and compact.*

Proof. The convexity of $\mathbf{G}_c^1$ follows from the fact that $\mathbf{G}_c$ is closed under addition and multiplication by positive numbers. Since $\mathbf{G}_c$ is also closed under passages to the limit, to prove that $\mathbf{G}_c^1$ is compact it suffices to show that it is conditionally compact.

Put for $x \in [0,\infty)$

$$d(x) = \sup\{ M([x,\infty)) : M \in \mathbf{G}_c^1 \}.$$

This function, being non-increasing, tends to a limit $d(\infty)$ as $x \to \infty$. Given $1 < u < v$ we have $j(M,u) \leq u^2$, which yields

$$j(M,v) = j(M,u) + \int_u^{v-} y^2\, M(dy) \leq u^2 + v^2(M([u,\infty)) - M([v,\infty))).$$

Further, for any $M \in \mathbf{G}_c^1$ we have $M([v,\infty)) \leq i(M,v) \leq cv^{-2}j(M,v)$, which together with the previous inequality implies

$$M([v,\infty)) \leq cu^2v^{-2} + c(M([u,\infty)) - M([v,\infty))).$$

Using the above inequality we get

$$(1+c)d(v) \leq cu^2v^{-2} + cd(u).$$

Finally, letting $v \to \infty$ and then $u \to \infty$ we obtain $(1+c)d(\infty) \leq cd(\infty)$. Thus $d(\infty) = 0$, which shows that $\mathbf{G}_c^1$ is conditionally compact. This terminates the proof.

Since, by Lemma 2.3, $\mathbf{Q}_c^1$ is a closed subset of $\mathbf{G}_c^1$, we have, as a consequence of Proposition 3.1, the following statement.

PROPOSITION 3.2. *The set $\mathbf{Q}_c^1$ is compact.*

In what follows by $\operatorname{ext}\mathbf{Z}$ we shall denote the set of extreme points of a convex set $\mathbf{Z}$.

PROPOSITION 3.3. $\mathbf{Q}_c^1 \subset \operatorname{ext}\mathbf{G}_c^1$.

Proof. Suppose that $M \in \mathbf{Q}_c^1$. To prove that $M \in \operatorname{ext}\mathbf{G}_c^1$ it suffices to show that for any pair $M_1, M_2 \in \mathbf{G}_c$ fulfilling the equation $M_1+M_2 = M$ the measure M_1 is proportional to M. One can easily obtain from this equation the formulae $S(M) = S(M_1) \cup S(M_2)$ and $T(M,c) = T(M_1,c) \cap T(M_2,c)$, which, by (2.14), yield for $j = 1,2$

$$T(M,c) = T(M_j,c) = S(M_j) = S(M).$$

Thus $M_1, M_2 \in \mathbf{Q}_c$. Applying Proposition 2.2 we conclude that M_1 is proportional to M, which completes the proof.

LEMMA 3.1. *Let $M \in \mathbf{G}_c^1$. Suppose that $S(M) \setminus T(M,c)$ is non-empty and consists of points isolated in $S(M)$. Then $M \notin \operatorname{ext}\mathbf{G}_c^1$.*

Proof. The assumption that the points of $S(M) \setminus T(M,c)$ are isolated in $S(M)$ implies

$$S(M) \setminus T(M,c) \subset A(M). \tag{3.1}$$

Choose $v \in S(M)\setminus T(M,c)$. Of course $v > 0$ and $v \in A(M)$. Since $0 \in S(M)$ we can find $u \in S(M)$ fulfilling $0 \leq u < v$ and

$$(u,v) \cap S(M) = \emptyset. \tag{3.2}$$

The relation $u \in A(M)$ follows from Lemma 2.2 in the case $u \in T(M,c)$ and from (3.1) in the remaining case. Introduce the notation

$$a_1 = M(\{u\}), \quad a_2 = M(\{v\}),$$

$$a_3 = \int_0^v \frac{y^2}{m(y)} M(dy), \quad a_4 = \int_{v+}^{\infty} \frac{M(dy)}{m(y)}, \tag{3.3}$$

$$\left.\begin{aligned} b_1 &= (v^2-u^2)^{-1}(a_3u^2 + a_4u^2v^2) \\ b_2 &= (v^2-u^2)^{-1}(a_3v^2 + a_4u^2v^2) \end{aligned}\right\} \text{ if } u > 0, \tag{3.4}$$

$$b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2} + a_3 \quad \text{if } u = 0, \tag{3.5}$$

$$a_0 = c - l(M, v), \tag{3.6}$$

$$a = \min\left(\frac{1}{2}, \frac{1}{2}a_1 b_1^{-1}\frac{u^2}{m(u)}, a_2 b_2^{-1}\frac{v^2}{m(v)}, \right. \\ \left. a_0 j(M, v)(cb_1 + b_2 + a_4 v^2)^{-1}\right). \tag{3.7}$$

It is clear that the numbers $a_0, a_1, a_2, a_3, b_1, b_2$ and a are positive and $a_4 \geq 0$.

Define a pair M_1, M_2 of measures by setting for any Borel subset B of $[0, \infty)$ and $r = 1, 2$

$$M_r(B) = M(B) - (-1)^r a M(B \cap (v, \infty)) \\ + (-1)^r a b_1 \frac{m(u)}{u^2}\delta_u(B) - (-1)^r a b_2 \frac{m(v)}{v^2}\delta_v(B). \tag{3.8}$$

Taking into account (3.7) we get $M_r(B) \geq M(B \cap [0, u))$ if $u > 0$ and $M_r(B) \geq \frac{1}{2}a_1\delta_0(B)$ if $u = 0$. Hence $M_1, M_2 \in \mathbf{G}$.

Now we prove that $M_1, M_2 \in \mathbf{G}_c$. Observe that for $x \in (v, \infty)$

$$i(M_r, x) = (1 - (-1)^r a) i(M, x),$$
$$j(M_r, x) = (1 - (-1)^r a) j(M, x) + (-1)^r a(a_3 + b_1 - b_2),$$

which, by (3.4) and (3.5), yields

$$j(M_r, x) = (1 - (-1)^r a) j(M, x).$$

Thus

$$l(M_r, x) = l(M, x) \quad \text{for } x \in (v, \infty). \tag{3.9}$$

Further, by (3.2), we have for $x \in (u, v]$

$$i(M_r, x) = i(M, v) - (-1)^r a(a_4 + b_2 v^{-2}),$$
$$j(M_r, x) = j(M, v) + (-1)^r a b_1.$$

Thus, by (3.6),

$$x^2 i(M_r, x) \leq v^2 i(M_r, x) \leq v^2 i(M, v) + a(a_4 v^2 + b_2) \\ = (c - a_0) j(M, v) + a(a_4 v^2 + b_2)$$

and, by (3.7),

$$a(a_4 v^2 + b_2 + cb_1) \leq a_0 j(M, v),$$

which yields

$$x^2 i(M_r, x) \leq cj(M, v) - cab_1 \leq j(M_r, x).$$

This proves the inequality

$$l(M_r, x) \leq c \quad \text{for } x \in (u, v]. \tag{3.10}$$

In the case $u = 0$ formulae (3.9) and (3.10) prove the relation $M_1, M_2 \in \mathbf{G}_c$. Suppose that $u > 0$ and $x \in (0, u]$. Then $j(M_r, x) = j(M, x)$ and

$$i(M_r, x) = i(M, x) - (-1)^r a(a_4 - b_1 u^{-2} + b_2 v^{-2}),$$

which, by (3.4), yields $i(M_r, x) = i(M, x)$. Consequently, $l(M_r, x) = l(M, x)$ for $x \in (0, u]$. Comparing this with (3.9) and (3.10) we conclude that $M_1, M_2 \in \mathbf{G}_c$ in the case $u > 0$.

From (3.8) it follows immediately that $M = \frac{1}{2}(M_1 + M_2)$. Consequently, to prove that $M \notin \operatorname{ext} \mathbf{G}_c^1$ it suffices to show that M_1 is not proportional to M. Contrary to this, suppose that $M_1 = bM$ for a positive constant b. In particular, $M_1(\{u\}) = bM(\{u\})$ and $M_1(\{v\}) = bM(\{v\})$, which, by (3.3) and (3.8), yields $a_1 - ab_1 u^{-2} m(u) = a_1 b$, $a_2 + ab_2 v^{-2} m(v) = a_2 b$. From the first equation we get $b < 1$ and from the second one $b > 1$, which gives the required contradiction. The lemma is thus proved.

LEMMA 3.2. *Let $M \in \mathbf{G}_c^1$. Suppose that $S(M) \setminus T(M, c)$ is non-empty and contains at least one accumulation point of $S(M)$. Then $M \notin \operatorname{ext} \mathbf{G}_c^1$.*

Proof. Let u be an accumulation point of the set $S(M)$ belonging to $S(M) \setminus T(M, c)$. Of course, $u > 0$. We know that $T(M, c)$ is closed and $l(M, \cdot)$ is continuous from the left and has right-hand limits satisfying (2.9). Consequently, we can find u_1, u_2 with $0 < u_1 < u < u_2$ and

$$\sup\{\, l(M, x) : x \in (u_1, u_2] \,\} = c - a_0$$

where $a_0 > 0$. Of course $M((u_1, u_2]) > 0$. The interval $(u_1, u_2]$ contains an accumulation point of $S(M)$ and, consequently, can be divided into three subintervals $(v_1, v_2]$, $(v_2, v_3]$ and $(v_3, v_4]$ such that $u_1 = v_1 < v_2 < v_3 < v_4 = u_2$ and $M((v_n, v_{n+1}]) > 0$ $(n = 1, 2, 3)$. Introduce the notation

$$a_n = \int\limits_{v_n+}^{v_{n+1}} \frac{y^2}{m(y)} M(dy), \qquad a_{n+3} = \int\limits_{v_n+}^{v_{n+1}} \frac{M(dy)}{m(y)} \qquad (n = 1, 2, 3),$$

$$a_7 = \int\limits_0^{v_1} \frac{y^2}{m(y)} M(dy).$$

It is clear that $a_1, a_2, \ldots, a_7$ are all positive. Let b_1, b_2, b_3 be a solution to

$$(3.11) \qquad a_1 b_1 + a_2 b_2 + a_3 b_3 = 0, \qquad a_4 b_1 + a_5 b_2 + a_6 b_3 = 0,$$

not identically vanishing. Since all coefficients are positive, we conclude that $b_1 = b_2 = b_3$ is impossible. Without loss of generality we may assume that

$$(3.12) \qquad |b_n| < 1 \qquad (n = 1, 2, 3).$$

Put

$$a = \min\bigl(1, a_0 a_7(|b_1|(ca_1 + v_4^2 a_4) + |b_2|(ca_2 + v_4^2 a_5) + |b_3|(ca_3 + v_4^2 a_6))^{-1}\bigr). \tag{3.13}$$

It is clear that $a > 0$.

Define a pair M_1, M_2 of measures by setting for any Borel subset B of $[0, \infty)$ and $r = 1, 2$

$$M_r(B) = M(B) + \sum_{n=1}^{3} (-1)^r a b_n M(B \cap (v_n, v_{n+1}]). \tag{3.14}$$

From (3.12) and (3.13) it follows that $|ab_n| < 1$ $(n = 1, 2, 3)$. Consequently, $M_r(B) \geq M(B \cap [0, v_1))$, which shows that $M_r \in \mathbf{G}$.

Now we prove that $M_1, M_2 \in \mathbf{G}_c$. From (3.14) by simple computations we get for $x \in (0, v_1]$

$$i(M_r, x) = i(M, x) + (-1)^r a(a_4 b_1 + a_5 b_2 + a_6 b_3), \qquad j(M_r, x) = j(M, x)$$

and for $x \in (v_4, \infty)$

$$i(M_r, x) = i(M, x), \qquad j(M_r, x) = j(M, x) + (-1)^r a(a_1 b_1 + a_2 b_2 + a_3 b_3),$$

which, by (3.11), yields

$$l(M_r, x) = l(M, x) \qquad \text{for } x \in (0, v_1] \cup (v_4, \infty). \tag{3.15}$$

Suppose now that $x \in (v_1, v_4]$. Then

$$\begin{aligned} i(M_r, x) &\leq i(M, x) + a(a_4|b_1| + a_5|b_2| + a_6|b_3|), \\ j(M_r, x) &\geq j(M, x) - a(a_1|b_1| + a_2|b_2| + a_3|b_3|). \end{aligned}$$

Since $l(M, x) \leq c - a_0$ for $x \in (v_1, v_4]$, we have $x^2 i(M, x) \leq (c - a_0) j(M, x) \leq cj(M_r, x) + ca(a_1|b_1| + a_2|b_2| + a_3|b_3|) - a_0 a_7$. Consequently,

$$\begin{aligned} x^2 i(M_r, x) &\leq x^2 i(M, x) + av_4^2(a_4|b_1| + a_5|b_2| + a_6|b_3|) \\ &\leq cj(M_r, x) + c(a_1|b_1| + a_2|b_2| + a_3|b_3|) \\ &\quad + av_4^2(a_4|b_1| + a_5|b_2| + a_6|b_3|) - a_0 a_7, \end{aligned}$$

which, by (3.13), yields $x^2 i(M_r, x) \leq cj(M_r, x)$ for $x \in (v_1, v_4]$. This inequality together with (3.15) implies $M_r \in \mathbf{G}_c$.

Obviously, $M = \frac{1}{2}(M_1 + M_2)$. To prove that $M \notin \operatorname{ext} \mathbf{G}_c^1$ it suffices to show that M_1 is not proportional to M. The proportionality would imply

$$M_1((v_n, v_{n+1}]) = bM((v_n, v_{n+1}]) \qquad (n = 1, 2, 3)$$

for a positive constant b. Thus, by (3.14), we would have $b_1 = b_2 = b_3$, which is impossible. This terminates the proof.

The following equality is an immediate consequence of Proposition 3.3 and Lemmas 3.1 and 3.2.

PROPOSITION 3.4. $\operatorname{ext} \mathbf{G}_c^1 = \mathbf{Q}_c^1$.

4. An integral representation. Let $\mathcal{E}$ be the space of all closed subsets A of $[0, \infty)$ with $0 \in A$. Identifying the points 0 and ∞ we regard the half-line $[0,\infty)$ as a subset of a circle with the usual metric ρ. It is clear that $\mathcal{E}$ consists of all ρ-compact subsets $A \subset [0,\infty)$ with $0 \in A$. The metric ρ induces the Hausdorff distance ρ_H between subsets of $[0,\infty)$. It is easy to see that the space $\mathcal{E}$ with the metric ρ_H is compact. Moreover, by [6], Section 42,II, the topology induced by ρ_H coincides with the topology induced by the topological limit.

To prepare the way for obtaining an integral representation for measures from $\mathbf{G}_c$ we proceed to describe the measures belonging to $\operatorname{ext} \mathbf{G}_c^1$. Given $A \in \mathcal{E}$ we put $z(A) = \min(1, w(A))$ and for $c > 0$, $x \in (0, w(A)]$

$$g(A,c,x) = \frac{2cx}{cx^2 + n^2(A,x)} \,. \tag{4.1}$$

We define a function $h(A, c, \cdot)$ by setting

$$h(A,c,0) = 0, \tag{4.2}$$

$$h(A,c,x) = (1+c)^{-1} \exp \int_{z(A)}^{n(A,x)} g(A,c,y)\,dy \qquad \text{if } x \in (0, w(A)], \tag{4.3}$$

$$h(A,c,x) = \exp \int_{z(A)}^{w(A)} g(A,c,y)\,dy \qquad \text{if } x \in (w(A), \infty). \tag{4.4}$$

Since

$$g(A,c,y) = \frac{2cy}{cy^2 + n^2(A,x)} \qquad \text{for } y \in [x, n(A,x)],$$

we have, by (4.3),

$$h(A,c,x) = \frac{n^2(A,x)}{cx^2 + n^2(A,x)} \exp \int_{z(A)}^{x} g(A,c,y)\,dy \qquad \text{for } x \in (0, w(A)]. \tag{4.5}$$

Observe that, by (4.1), the inequality $n(A,x) \geq x$ yields

$$g(A,c,x) \leq 2c(1+c)^{-1}x^{-1} \qquad \text{for } x \in (0, w(A)].$$

Together with (4.2)–(4.4) this gives

$$h(A,c,x) \leq \max(1, x^{2c/(1+c)}) \qquad \text{for } x \in [0,\infty). \tag{4.6}$$

Hence it follows that

$$\lim_{x\to\infty} x^{-2}h(A,c,x) = 0. \tag{4.7}$$

Moreover, using (4.5) it is easy to check that the derivative of the function $\frac{1}{2}x^{-2}\exp\int_{z(A)}^{x} g(A,c,y)\,dy$ is equal to $-x^{-3}h(A,c,x)$ on $(0,w(A)]$. Thus

$$\int_{x}^{\infty} y^{-3}h(A,c,y)\,dy = \frac{1}{2}x^{-2}\exp\int_{z(A)}^{x} g(A,c,y)\,dy \tag{4.8}$$

$$\text{for } x\in(0,w(A)].$$

Further, from (4.4) we get

$$\int_{x}^{\infty} y^{-3}h(A,c,y)\,dy = \frac{1}{2}x^{-2}\exp\int_{z(A)}^{w(A)} g(A,c,y)\,dy \tag{4.9}$$

$$\text{for } x\in(w(A),\infty).$$

Observe that by the definitions (4.2)–(4.4) and formula $n(A,w(A)) = w(A)$ the function $h(A,c,\cdot)$ is non-decreasing and continuous from the left on $[0,\infty)$. Consequently, there exists a Borel measure H_A^c finite on every bounded subset of $[0,\infty)$ such that

$$H_A^c([0,x)) = h(A,c,x) \quad \text{for } x\in(0,\infty). \tag{4.10}$$

Since A coincides with the set of points of increase of $h(A,c,\cdot)$, we infer that

$$S(H_A^c) = A. \tag{4.11}$$

Put for any Borel subset B of $[0,\infty)$

$$M_A^c(B) = \int_B \frac{m(y)}{y^2}\,H_A^c(dy). \tag{4.12}$$

Evidently, by (4.11),

$$S(M_A^c) = A. \tag{4.13}$$

Moreover, by (4.10),

$$M_A^c([0,x)) = h(A,c,x) \quad \text{for } x\in(0,1]. \tag{4.14}$$

Integrating by parts the right-hand side of (4.12) and applying (4.10) we get for $x\in(1,\infty)$

$$\begin{aligned} M_A^c([0,x)) &= M_A^c([0,1)) + \int_1^x y^{-2}\,H_A^c(dy) \\ &= x^{-2}h(A,c,x) + 2\int_1^x y^{-3}\,h(A,c,y)dy. \end{aligned}$$

Using (4.5), (4.8) and (4.9) we obtain

(4.15) $\quad M_A^c([0,x)) = 1 \quad$ if $w(A) \le 1$ and $x \in (1,\infty)$,

(4.16) $\quad M_A^c([0,x)) = 1 - cn^{-2}(A,x)h(A,c,x)$
if $w(A) > 1$ and $x \in (1, w(A)]$,

(4.17) $\quad M_A^c([0,x)) = 1 \quad$ if $w(A) > 1$ and $x \in (w(A),\infty)$.

Hence, by (4.7), M_A^c is a probability measure. Observe that, by (4.10) and (4.12),

$$j(M_A^c, x) = h(A,c,x) \qquad \text{for } x \in (0,\infty).$$

Moreover, by (2.5) and (4.8),

$$k(M_A^c, x) = \frac{1}{2}x^{-2} \exp \int_{z(A)}^{x} g(A,c,y)\,dy \qquad \text{for } x \in (0, w(A)],$$

which, by (2.6) and (4.5), yields

$$i(M_A^c, x) = cn^{-2}(A,x)h(A,c,x) \qquad \text{for } x \in (0, w(A)].$$

The equality $i(M_A^c, x) = 0$ for $x \in (w(A),\infty)$ is evident. Thus

$$l(M_A^c, x) = cn^{-2}(A,x)x^2 \qquad \text{for } x \in (0, w(A)]$$

and $l(M_A^c, x) = 0$ otherwise. This proves that $M_A^c \in \mathbf{G}_c^1$ and $T(M_A^c, c) = A$, which, by (4.13), implies $M_A^c \in \mathbf{Q}_c^1$. Applying Proposition 2.2 we conclude that

$$\mathbf{Q}_c^1 = \{\, M_A^c : A \in \mathcal{E} \,\}.$$

The lower semicontinuity of $M \to S(M)$ and the upper semicontinuity of $M \to T(M,c)$ on $\mathbf{G}_c$ yield, by (2.14), the continuity of the function $M_A^c \to S(M_A^c) = A$. Since the mapping $M_A^c \to A$ from $\mathbf{Q}_c^1$ onto $\mathcal{E}$ is one-to-one and both spaces $\mathbf{Q}_c^1$ and $\mathcal{E}$ are compact, the inverse mapping $A \to M_A^c$ is also continuous. Thus, by Proposition 3.4, we have the following description of extreme points.

PROPOSITION 4.1. *For any $c > 0$*

$$\operatorname{ext} \mathbf{G}_c^1 = \{\, M_A^c : A \in \mathcal{E} \,\}.$$

The mapping $A \to M_A^c$ is a homeomorphism between $\mathcal{E}$ and ext $\mathbf{G}_c^1$.

In attempting to visualize the extreme points of $\mathbf{G}_c^1$ we shall give some examples.

1. $A_1 = \{0\}$. Then $M_{A_1}^c = \delta_0$ and $e(M_{A_1}^c)$ is the standard Gaussian measure.

2. $A_2 = [0,\infty)$. In this case $h(A_2,c,x) = (1+c)^{-1}x^{2c/(1+c)}$ for $x \in (0,\infty)$, which, by (4.14) and (4.16), yields

$$M^c_{A_2}([0,x)) = (1+c)^{-1}x^{2c/(1+c)} \qquad \text{for } x \in (0,1],$$
$$M^c_{A_2}([0,x)) = 1 - c(1+c)^{-1}x^{-2/(1+c)} \qquad \text{for } x \in (1,\infty).$$

Moreover,

$$\hat{e}(M^c_{A_2})(t) = \exp(-b|t|^{2/(1+c)})$$

where b is a positive constant. Thus $e(M^c_{A_2})$ is a symmetric stable measure with exponent $2/(1+c)$.

3. $A_3 = \{0\} \cup \{d^k : k = 0, \pm1, \pm2, \ldots\}$ where $d > 1$. Then

$$h(A_3,c,x) = (1+c)^{k-1}d^{2k}(d^2+c)^{-k} \qquad \text{if } x \in (d^{k-1}, d^k].$$

By simple calculations we get

$$\hat{e}(M^c_{A_3})(t) = \exp a \sum_{k=-\infty}^{\infty} (\cos td^k - 1)b^k$$

where $a = c(d^2-1)(1+c)^{-1}(d^2+c)^{-1}$ and $b = (1+c)(d^2+c)^{-1}$. Thus the measure $e(M^c_{A_3})$ is quasi-stable in the sense of Kruglov [4].

4. $A_4 = \{0\} \cup \{w\}$ where $w > 0$. Then

$$M^c_{A_4} = z^2(A_4)(z^2(A_4)+c)^{-1}\delta_0 + c(z^2(A_4)+c)^{-1}\delta_w.$$

5. $A_5 = [0,1]$. Then $M^c_{A_5}([0,x)) = (1+c)^{-1}x^{2c/(1+c)}$ for $x \in (0,1]$ and $M^c_{A_5}([0,x)) = 1$ otherwise.

PROPOSITION 4.2. *For any $c > 0$ the set of atoms of the measure M^c_A consists of the point $w(A)$ provided $w(A) < \infty$ and of the points of discontinuity of the function $n(A,\cdot)$ contained in the interval $[0,w(A))$.*

Proof. By (4.5), (4.10) and (4.12) we conclude that the set of atoms of M^c_A belonging to $[0,w(A))$ coincides with the set of points of discontinuity of $n(A,\cdot)$ in $[0,w(A))$. Suppose that $w(A) < \infty$. Then, by (4.3), (4.4) and (4.10), $H^c_A([0,w(A)]) = (1+c)H^c_A([0,w(A)))$, which shows that $H^c_A(\{w(A)\}) > 0$. Consequently, $M^c_A(\{w(A)\}) > 0$. This completes the proof.

LEMMA 4.1. *For any $A \in \mathcal{E}$, $c > 0$ and $p \in (0,2/(1+c))$*

$$\int_0^{\infty} x^p\, M^c_A(dx) \le (2+pc^2-p)(2-pc-p)^{-1}. \tag{4.18}$$

Proof. Suppose that $p \in (0,2/(1+c))$. In the case $w(A) \le 1$ the inequality

$$\int_0^\infty x^p \, M_A^c(dx) \le 1$$

is obvious. Since the right-hand side of (4.18) is greater than 1, we have the desired inequality. Suppose now that $w(A) > 1$. Taking into account the inequality $n(A,x) \ge x$ for $x \in (0, w(A)]$ and formulae (4.6) and (4.16) we get

$$M_A^c([x,\infty)) \le cx^{-2/(1+c)} \quad \text{for } x \in (1,\infty). \tag{4.19}$$

Thus

$$\lim_{x\to\infty} x^p M_A^c([x,\infty)) = 0.$$

Making use of this relation and integrating by parts we have

$$\int_0^\infty x^p \, M_A^c(dx) \le 1 + p \int_1^\infty x^{p-1} \, M_A^c([x,\infty))dx,$$

which, by (4.19), yields (4.18). This terminates the proof.

Applying the Choquet Theorem [1] on extreme points and making use of Proposition 4.1 we get the main result of this paper.

THEOREM 4.1. *A measure M belongs to the family $\mathbf{G}_c$ if and only if it is of the form*

$$M = \int_{\mathcal{E}} M_A^c \, \Lambda(dA)$$

where Λ is a finite Borel measure on $\mathcal{E}$ which does not vanish identically and the integral is taken in the weak sense.

One of the useful conclusions we can draw from the above theorem is that the measures from $\mathbf{F}_{sym}$ have moments of sufficiently small order.

THEOREM 4.2. *For any $p \in (0, 2/(1+c))$ and $M \in \mathbf{G}_c$ the p-th moment of the measure $e(M)$ is finite.*

Proof. From Theorem 4.1 and Lemma 4.1 it follows that $\int_0^\infty x^p \, M(dx)$ is finite. Applying Kruglov's Lemma ([4], Lemma 1) we get the desired assertion.

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Reçu par la Rédaction le 8.11.1989