COLLOQUIUM MATHEMATICUM

VOL. XXII 1970 FASC. 1

# A PROOF OF THE MOORE METRIZATION THEOREM

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**1.** Several references to the little-known Moore metrization theorem have appeared recently in [2], [3], and [4], the most recent being in the important paper of Roy [6]. It appears that under certain general circumstances, this theorem is much more natural to apply than either the Nagata-Smirnov or Bing metrization theorems. Since a direct proof of this result is not explicitly available in the literature, we would like to present one based on an early theorem of Alexandroff and Urysohn [1] and a recent alternative proof of this theorem by Rolfsen [5].

**2.** Let $S$ be a topological space and $H$ a collection of subsets of $S$. If $A \subseteq S$, then $\mathrm{St}(A, H)$ means the union of all those members of $H$ which meet $A$. If $x \in S$, then $\mathrm{St}(x, H)$ means $\mathrm{St}(\{x\}, H)$. A *development* for $S$ is a countable family $G_1, G_2, \ldots$ of collections of open sets such that, for each $i$, $G_i$ covers $S$, and if $x \in U$, where $U$ is open, there is an $i$ such that $x \in \mathrm{St}(x, G_i) \subseteq U$.

A development $G_1, G_2, \ldots$ for $S$ is termed *regular* if, for each $i$, whenever two members of $G_{i+1}$ intersect, their union is a subset of some member of $G_i$. A development $G_1, G_2, \ldots$ for $S$ is termed *neighborhood-star* if for each open set $U$ and $x \in U$, there exists an $i$ and a neighborhood $N$ of $x$ such that $\mathrm{St}(N, G_i) \subseteq U$.

THEOREM 1 (Alexandroff-Urysohn). *A topological space is metrizable if and only if it is Hausdorff and admits a regular development.*

Rolfsen [5] was able to show that a topological space admits a regular development if and only if it admits a neighborhood-star development. Consequently, Theorem 1 above yields:

THEOREM 2. *A topological space is metrizable if and only if it is Hausdorff and admits a neighborhood-star development.*

**3.** THEOREM 3 (Moore metrization theorem). *A topological space $S$ is metrizable if and only if there exists a countable collection $G_1, G_2, \ldots$ such that*

(1) *Each* $G_i$ *is an open cover of* $S$.

(2) *Given distinct points,* $x$ *and* $y$, *in* $S$, *and any open set* $U$, *containing* $x$, *there exists an* $i$ *such that if* $V$ *and* $W$ *are elements of* $G_i$, $x \in V$, *and* $V \cap W \neq \emptyset$, *then* $W \subseteq U$ *and* $y \notin W$.

Proof. Suppose $S$ is metrizable, with metric $d$. For each positive integer $i$, let $G_i = \{S(x, 2^{-(i+2)}) \mid x \in S\}$, where $S(x, r)$ means $\{y \in S \mid d(x, y) < r\}$. Now let $x$ and $y$ be distinct points of $S$, and let $U$ be an open set containing $x$. Let $i_1$ be such that $d(x, y) > 2^{-i_1}$. Let $i_2$ be such that $S(x, 2^{-i_2}) \subseteq U$. Let $i' = \max\{i_1, i_2\}$. Consider the collection $G_{i'}$. If $V$ and $W$ belong to $G_{i'}$, $x \in V$, and $V \cap W \neq \emptyset$, then $\sup\{d(a, b) \mid a, b \in V \cup W\} < 2^{-i'}$, so that, for every $a \in W$, $d(a, x) < 2^{-i'} \leqslant 2^{-i_2}$. This means that $W \subseteq U$. Moreover, since $d(x, y) > 2^{-i_1} \geqslant 2^{-i'}$, $y \notin W$.

Suppose now there exists a countable collection $G_1, G_2, \ldots$ satisfying conditions (1) and (2) of the theorem. We will show that $S$ is Hausdorff and that the collection $G_1, G_2, \ldots$ is a neighborhood-star development for $S$. By Theorem 2, $S$ will then be metrizable.

Let $U$ be an open set and $x \in U$. Then there is an $i$ such that if $V$ and $W$ belong to $G_i$, $x \in V$, and $V \cap W \neq \emptyset$, then $W \subseteq U$. Since $G_i$ covers $S$, $x \in V_1$ for some $V_1 \in G_i$, and, by the above, $\mathrm{St}(V_1, G_i) \subseteq U$. Thus, $G_1, G_2, \ldots$ is a neighborhood-star development.

Now let $x$ and $y$ be distinct elements of $S$. Then, for some $i$, condition (2) is satisfied. Let $V$ and $W$ be elements of $G_i$ such that $x \in V$ and $y \in W$. Then $V \cap W = \emptyset$ and so $S$ is Hausdorff.

## *REFERENCES*


[1] P. Alexandroff and P. Urysohn, *Une condition nécessaire et suffisante pour qu'une classe (L) soit une classe (B)*, Comptes Rendus de l'Académie des Sciences 177 (1923), p. 1274-1276.

[2] R. W. Heath, *Arcwise connectedness in semi-metric spaces*, Pacific Journal of Mathematics 12 (1962), p. 1301-1319.

[3] F. B. Jones, *Metrization*, American Mathematical Monthly 73 (1966), p. 571-575.

[4] L. F. McAuley, *A relation between perfect separability, completeness, and normality in semi-metric spaces*, Pacific Journal of Mathematics 6 (1956), p. 315-326.

[5] D. Rolfsen, *Alternative metrization proofs*, Canadian Journal of Mathematics 18 (1966), p. 750-757.

[6] P. Roy, *Nonequality of dimensions for metric spaces*, Transactions of the American Mathematical Society 134 (1968), p. 117-132.


*Reçu par la Rédaction le 30. 7. 1969*