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UNITARY PARTS OF GENERALIZED HANKEL OPERATORS

BY

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1. Introduction. Let T be a contractive linear operator on a separable complex Hilbert space X. Such an operator is said to be *completely non-unitary* if it has no non-trivial reducing subspace N such that the restriction $T|N$ of T to N is unitary. It is known (see [7], Theorem I.3.2) that for any contraction T on X we can find a unique orthogonal decomposition $X = M \oplus M_1$ such that M and M_1 reduce T, $T|M$ is unitary, and $T|M_1$ is completely non-unitary. It is not excluded that M or M_1 is possibly the subspace $\{0\}$. Furthermore, M is given by

$$M = \{x \in X: \|T^n x\| = \|x\| = \|T^{*n} x\|, n = 1, 2, \ldots\}$$

and is called the *unitary subspace* of T. The restriction $T|M$ is called the *unitary part* of T.

In the following, we describe explicitly the above decomposition for the class of generalized Hankel operators.

2. Generalized Toeplitz and Hankel operators. We say that an isometry $S: X \to X$ is a *unilateral shift* if there exists a subspace C in X for which $(S^j C) \perp (S^k C)$ for non-negative integers $j \neq k$ and

$$X = \bigoplus_{n=0}^{\infty} S^n C.$$

It is not difficult to see that C is uniquely determined by S, viz., $C = (SX)^{\perp}$.

In what follows, we shall find it convenient to consider the minimal unitary extension U of S. Then U is a bilateral shift acting on a Hilbert space Y containing X, and Y decomposes into the orthogonal direct sum

$$Y = \bigoplus_{n=-\infty}^{\infty} U^n C \quad \text{and} \quad U|X = S.$$

That such a space Y exists is clear, since it can be constructed as a direct sum of a countable number of copies of C indexed on the integers. Equivalently, U and Y are obtained as the minimal unitary dilation of S in the structure theory of Sz.-Nagy and Foiaş [7], Chapter I.

We now fix a unilateral shift S on the Hilbert space X and make the following definitions.

Definitions. 1. A bounded operator $T\colon X \to X$ is *Toeplitz* if $S^*TS = T$.

2. A bounded operator $H\colon X \to X$ is *Hankel* if $HS = S^*H$.

3. A bounded operator $L\colon Y \to Y$ is *Laurent* if $LU = UL$.

These operators have previously been studied by Rosenblum [6] and Page [4], and they are generalizations of the classical Toeplitz and Hankel matrices $T = (c_{j-k})$ and $H = (c_{j+k})$, $j, k = 0, 1, 2, \ldots$, acting on l^2. In particular, we have the following two facts which, in this generalized setting, have been observed by Page [4], Theorems 1 and 2:

1. *A bounded operator T on X is Toeplitz if and only if there exists a bounded Laurent operator L on Y such that $T = P_+L\,|\,X$, where $P_+\colon Y \to X$ is the orthogonal projection. In this case, $\|L\| = \|T\|$.*

2. *A bounded operator H on X is Hankel if and only if there is a bounded operator J on Y satisfying $U^*J = JU$ such that $H = P_+J\,|\,X$. Also, J can be chosen so that $\|J\| = \|H\|$.*

The results above assert that every bounded Toeplitz operator is obtained by projection onto X of an operator that commutes with U, and that every bounded Hankel operator may be obtained by projection onto X of an operator J satisfying $U^*J = JU$. The latter result is an analogue for generalized Hankel operators of a well-known theorem of Nehari [3] giving criteria for a classical Hankel matrix to be a bounded operator on l^2.

In the generalized framework described above, we can give descriptions of the unitary parts of Toeplitz and Hankel operators. In doing so, we shall use the fact that the spaces X and Y are isometrically isomorphic to certain function-space models obtained as follows.

Let m denote the normalized Lebesgue measure on $[0, 2\pi]$. Then L^2_C denotes the Hilbert space of weakly measurable functions defined on the unit circle, taking values in the Hilbert space C, and having square-integrable norms. If $\langle\cdot, \cdot\rangle$ denotes the inner product in L^2_C and $(\cdot, \cdot)$ that in C, then

$$\langle x, y\rangle = \int_0^{2\pi} \big(x(e^{i\theta}), y(e^{i\theta})\big)\, dm(\theta) \qquad \text{for } x, y \in L^2_C.$$

A function $x \in L^2_C$ has a Fourier expansion of the form

$$x(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \qquad \text{with } c_n \in C,$$

such that for every $d \in C$ we have the ordinary Fourier expansion

$$\left(x(e^{i\theta}), d\right) = \sum_{n=-\infty}^{\infty} (c_n, d) e^{in\theta}.$$

The Hardy space H_C^2 consists of those functions in L_C^2 whose Fourier coefficients vanish for $n < 0$. The isometric isomorphism referred to above is then given by the mapping

$$\sum_{n=-\infty}^{\infty} U^n c_n \mapsto \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

from Y onto L_C^2 which takes X onto H_C^2. Under this correspondence, the operator L on Y is unitarily equivalent to multiplication on L_C^2 by a function $f(e^{i\theta}) \in L^\infty\left(B(C)\right)$, the algebra of essentially bounded functions from the unit circle to the set $B(C)$ of bounded operators on the Hilbert space C.

3. Unitary parts. In previous work by the author (see [1] and [2]), explicit descriptions have been given for the unitary parts of a classical Hankel matrix and of a generalized Toeplitz operator. These results may be summarized as follows:

1. *Let H be a Hankel contraction on H^2. Then a necessary and sufficient condition for H to have a non-trivial unitary subspace M is that there exist a complex constant k, $|k| = 1$, such that $\bar{k}H$ is real (hence, self-adjoint) and that the subspace*

$$M_0 = \{x \in H^2 \colon \bar{k}Hx = x\} \oplus \{x \in H^2 \colon \bar{k}Hx = -x\}$$

satisfy $M_0 \neq \{0\}$. In such a case, $M = M_0$.

2. *Let T be a generalized Toeplitz contraction on the Hilbert space*

$$X = \bigoplus_{n=0}^{\infty} S^n C.$$

Then a necessary and sufficient condition for T to have a non-trivial unitary subspace M is that there exist a decomposition $C = C_0 \oplus C_1$, $C_0 \neq \{0\}$, for which

$$M = \bigoplus_{n=0}^{\infty} S^n C_0 \quad \text{and} \quad M^\perp = \bigoplus_{n=0}^{\infty} S_n C_1,$$

and that there exist a unitary operator $R_0 \colon C_0 \to C_0$ such that

$$Tx = \sum_{n=0}^{\infty} S^n R_0 c_n \quad \text{for } x = \sum_{n=0}^{\infty} S^n c_n \in M.$$

We are now able to obtain criteria for the existence of a non-trivial unitary part of a generalized Hankel contraction analogous to that indicated above for classical Hankel matrices. The proof of the previous result uses the fact that if a Toeplitz contraction on H^2 has a non-trivial unitary part, then it must be a unitary operator on all of H^2, equal to multiplication by a constant of modulus 1. Since a generalized Toeplitz operator T on a Hilbert space X can have a non-trivial unitary subspace N_0 which is properly contained in X with $T\,|\,N_0$, not necessarily multiplication by a constant, we expect the self-adjoint operator $\bar{k}H\colon H^2 \to H^2$ to be replaced by a self-adjoint operator of the form $H\hat{K}_0^*$, acting on N_0. The next theorem shows that this is in fact the case.

Using the notation established above, for a Hankel operator H on H_C^2 $(\approx \bigoplus_{n=0}^{\infty} S^n C)$ we let the corresponding lifted operator $J\colon L_C^2 \to L_C^2$ for which $\|J\| = \|H\|$ and $H = P_+ J\,|\,H_C^2$ be given by $Jx = f(e^{i\theta})x(e^{-i\theta})$ for $f \in L^\infty(B(C))$.

THEOREM. *Let H be a generalized Hankel contraction on the Hilbert space*

$$X = \bigoplus_{n=0}^{\infty} S^n C.$$

Then necessary and sufficient conditions for H to have a non-trivial unitary subspace M are the following:

(i) *There exist decompositions $C = C_0 \oplus C_1$ and $X = N_0 \oplus N_1$ such that*

$$N_0 = \bigoplus_{n=0}^{\infty} S^n C_0 \quad \text{and} \quad N_1 = \bigoplus_{n=0}^{\infty} S^n C_1.$$

(ii) *There exists a unitary operator $K_0\colon C_0 \to C_0$ such that the corresponding operator $\hat{K}_0$ defined on N_0 by*

$$\hat{K}_0\Big(\sum_{n=0}^{\infty} S^n c_n\Big) = \sum_{n=0}^{\infty} S^n K_0 c_n$$

commutes with $S\,|\,N_0$, and such that $H\hat{K}_0^*$ is self-adjoint on M.

(iii) *The subspace*

$$M_0 = \{x \in N_0\colon H\hat{K}_0^* x = x\} \oplus \{x \in N_0\colon H\hat{K}_0^* x = -x\}$$

is not the zero subspace.

In such a case, $M = M_0$.

Proof. Suppose that H has a non-trivial unitary subspace M. Then there exists some $x \neq 0$ in X such that for H and its adjoint we have $\|H^n x\| = \|x\| = \|H^{*n} x\|$ for $n = 1, 2, \ldots$ Considering

$$x = \sum_{n=0}^{\infty} S^n c_n$$

as

$$x(e^{i\theta}) = \sum_{n=0}^{\infty} c_n e^{in\theta}$$

in H_C^2 with H given by $Hx = P_+ f(e^{i\theta})x(e^{-i\theta})$, $f \in L^\infty(B(C))$, we get for $n = 1$

$$\|x\| = \|Hx\| = \|P_+ f(e^{i\theta})x(e^{-i\theta})\| \leqslant \|f(e^{i\theta})x(e^{-i\theta})\| \leqslant \|x(e^{-i\theta})\|.$$

This implies

$$P_+ f(e^{i\theta})x(e^{-i\theta}) = f(e^{i\theta})x(e^{-i\theta}),$$

so that $Hx = f(e^{i\theta})x(e^{-i\theta}) \in H_C^2$.

Taking now $n = 2$, we get

$$\|x\| = \|H^2 x\| = \|Hf(e^{i\theta})x(e^{-i\theta})\| = \|P_+ f(e^{i\theta})f(e^{-i\theta})x(e^{i\theta})\|$$
$$\leqslant \|f(e^{i\theta})f(e^{-i\theta})x(e^{i\theta})\| \leqslant \|x(e^{i\theta})\|,$$

so that

$$H^2 x = f(e^{i\theta})f(e^{-i\theta})x(e^{i\theta}) = Tx,$$

where T is the generalized Toeplitz operator with symbol $\varphi(e^{i\theta}) = f(e^{i\theta})f(e^{-i\theta}) \in L^\infty(B(C))$.

Continuing, for $n = 0, 1, 2, \ldots$ we obtain

$$H^{2n+1}x(e^{i\theta}) = f(e^{i\theta})[f(e^{i\theta})f(e^{-i\theta})]^n x(e^{-i\theta}), \tag{1}$$

$$H^{2n}x(e^{i\theta}) = [f(e^{i\theta})f(e^{-i\theta})]^n x(e^{i\theta}). \tag{2}$$

Then (2) and its analogue for H^* imply that x is in the unitary subspace N_0 of the Toeplitz operator T. Therefore, by the above-mentioned result, we get the decompositions

$$C = C_0 \oplus C_1, \quad N_0 = \bigoplus_{n=0}^{\infty} S^n C_0, \quad N_1 = N_0^\perp = \bigoplus_{n=0}^{\infty} S^n C_1.$$

This gives the decomposition required by (i).

Also, by the previous result for Toeplitz operators there exists a unitary operator R_0: $C_0 \to C_0$ such that

$$T\Big(\sum_{n=0}^{\infty} S^n c_n\Big) = \sum_{n=0}^{\infty} S^n R_0 c_n.$$

If the operator R_0 has a spectral representation

$$R_0 = \int_0^{2\pi} e^{i\varphi} dE_\varphi,$$

then we let $K_0\colon C_0 \to C_0$ be the unitary operator given by

$$K_0 = \int_0^{2\pi} e^{i\varphi/2} dE_\varphi.$$

We make the natural extension of K_0 to an operator $\hat{K}_0$ on

$$N_0 = \bigoplus_{n=0}^{\infty} S^n C_0$$

by defining

$$\hat{K}_0\Big(\sum_{n=0}^{\infty} S^n c_n\Big) = \sum_{n=0}^{\infty} S^n K_0 c_n.$$

Both $\hat{K}_0$ and $\hat{K}_0^*$ commute with any operator that commutes with T (or T^*). (For $\hat{K}_0^*$ and T^*, we use a well-known theorem of Fuglede (see [5], Theorem 1.6.1) which states that if N is a bounded normal operator and B is a bounded operator such that $NB = BN$, then $N^*B = BN^*$.) In particular, $S\hat{K}_0 = \hat{K}_0 S$ and $H\hat{K}_0^* = \hat{K}_0^* H$, since

$$e^{i\theta}[f(e^{i\theta})f(e^{-i\theta})] = [f(e^{i\theta})f(e^{-i\theta})]e^{i\theta} \quad \text{and} \quad HT = H^3 = TH.$$

Furthermore, it is easy to see that the unitary subspace M of H reduces $H\hat{K}_0^*$ and that on this subspace we have $H^2 = T = \hat{K}_0^2$. Multiplying on the left by $\hat{K}_0^*$ and on the right by H^*, we obtain $\hat{K}_0^* H = \hat{K}_0 H^*$, and hence $H\hat{K}_0^* = \hat{K}_0 H^*$, which shows that the bounded operator $H\hat{K}_0^*$ is self-adjoint on M. This gives (ii).

Finally, the direct sum of the eigenspaces of $H\hat{K}_0^*$ for the eigenvalues $\lambda = \pm 1$, given by M_0, must be equal to the unitary subspace M of H since, clearly, $M_0 \subseteq M$ and, if $x \in M$, then

$$(H\hat{K}_0^*)^2 x = H^2 \hat{K}_0^{*2} x = TT^* x = x,$$

giving

$$x = \tfrac{1}{2}(x + H\hat{K}_0^* x) + \tfrac{1}{2}(x - H\hat{K}_0^* x) \in M_0.$$

Since these arguments are reversible, conditions (i)-(iii) give both necessary and sufficient conditions for H to have a non-trivial unitary subspace. The proof is then complete.

A simple, but illustrative, example of the decomposition above is given by taking $X = H^2 \oplus H^2$ with $H\colon X \to X$ defined by $H = H_0 \oplus H_1$, where $H_0\colon H^2 \to H^2$ is the Hankel operator with symbol $f(e^{i\theta}) = e^{i\theta}$ and $H_1\colon H^2 \to H^2$ is a completely non-unitary Hankel operator. In this case, H has a non-trivial unitary subspace given by

$$\begin{aligned} M &= \{x(e^{i\theta}) = (x_0, 0) + (x_1, 0)e^{i\theta}\colon x_0, x_1 \in C\} \\ &= \{x \in H^2 \oplus (0)\colon Hx = x\} \oplus \{x \in H^2 \oplus (0)\colon Hx = -x\}. \end{aligned}$$

It is the assertion of the theorem above that all abstract Hankel contractions decompose in a similar way.

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