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SOME ORDER THEORETIC CHARACTERIZATIONS OF THE 3-CELL

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We use* the work of Dyer and Hamstrom [5] on completely regular mappings to show that if a compact 3-manifold (resp. metric space) admits a certain type of partial order, then it is a 3-cell. In Section 3 we obtain some similar results using convex metrics. In the last two sections, we prove a fixed point theorem for partially ordered spaces and give a sufficient condition for a partially ordered space to be the product of an arc and a compact set.

1. Introduction. A *partially ordered space* is a compact, metric space X with a partial order $\leqslant$ such that $\leqslant$ is a closed subset of $X \times X$. In other words, if the sequences x_i and y_i converge in X to x and y, respectively, and $x_i \leqslant y_i$ for each i, then $x \leqslant y$.

For $x \in X$ we let

$$L(x) = \{y \in X \mid y \leqslant x\} \quad \text{and} \quad M(x) = \{y \in X \mid x \leqslant y\}.$$

For $A \subset X$ we let

$$L(A) = \cup\{L(x) \mid x \in A\} \quad \text{and} \quad M(A) = \cup\{M(x) \mid x \in A\}.$$

A *chain* is a totally ordered set. An *order arc* is a compact and connected chain. A separable and non-degenerate order arc is homeomorphic under an order preserving map to the unit interval $[0, 1]$ (where $[0, 1]$ has its usual order).

We let

$$\mathrm{Max}(X) = \{y \in X \mid M(y) = \{y\}\}$$

and

$$\mathrm{Min}(X) = \{y \in X \mid L(y) = \{y\}\}.$$

It is known [8] that each chain in X is contained in a maximal chain. Each maximal chain is closed and meets both $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$.

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A metric ϱ for X is *radially convex* if $a < b < c$ implies $\varrho(a, c) = \varrho(a, b) + \varrho(b, c)$. The following theorem about radially convex metrics is due to Carruth [4]:

THEOREM 1 (Carruth). *Every compact, metric, partially ordered space admits a radially convex metric.*

Definition. A mapping f of a metric space X onto a metric space Y is said to be a *0-regular mapping* provided f is open and if $y \in Y$, $p \in f^{-1}(y)$ and U is a neighbourhood of p, then there exists a neighbourhood N of p such that if $x \in Y$ and $a, b \in f^{-1}(x) \cap N$, then there is an arc from a to b in $f^{-1}(x) \cap U$.

We shall need the following special case of a theorem of Dyer and Hamstrom [5]:

THEOREM 2 (Dyer and Hamstrom). *Let F be a 0-regular mapping of a complete metric space X onto $[0, 1]$ such that, for each $y \in [0, 1]$, $F^{-1}(y)$ is homeomorphic to the point set K, where K is an i-cell or i-sphere ($i \leqslant 2$). Then there is a homeomorphism h of X onto the direct product $[0, 1] \times K$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & [0,1]\times K \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & [0,1] & \end{array}$$

commutes (where φ is the natural projection).

Most of the notation that we have not specifically defined is taken from Wilder [12].

2. Characterizations of the 3-cell.

THEOREM 3. *Let X be a compact 3-manifold with boundary S^2 and let $\theta \in X \setminus S^2$. If X admits a closed partial order such that* $\mathrm{Max}(X) = S^2$, $\mathrm{Min}(X) = \{\theta\}$ *and such that $L(x)$ is an order arc for each $x \in X$, then X is a 3-cell.*

Proof. Let Y be the quotient space obtained from X by collapsing $\mathrm{Max}(X)$ to a point and let π be the natural projection of X onto Y. Define a partial order $\leqslant^*$ on Y by letting $a \leqslant^* b$ in Y if and only if there exist $a' \in \pi^{-1}(a)$ and $b' \in \pi^{-1}(b)$ with $a' \leqslant b'$. Then Y is a partially ordered space.

By Theorem 1 there is a radially convex metric ϱ for Y such that

$$\varrho\big(\pi(\theta), \pi(\mathrm{Max}(X))\big) = 1.$$

Define a continuous function F from X onto $[0, 1]$ by letting

$$F(x) = \varrho\big(\pi(\theta), \pi(x)\big)$$

for each $x \in X$. For each $a \in [0, 1]$

$$F^{-1}(a) = \big\{x \in X \mid \varrho\big(\pi(\theta), \pi(x)\big) = a\big\}.$$

For each $m \in \mathrm{Max}(X)$, $F^{-1}(a)$ meets $L(m)$ in exactly one point.

If $a \leqslant b$ in $[0, 1]$, define a function g from $F^{-1}(b)$ onto $F^{-1}(a)$ by letting $g(x)$ be the unique point in $F^{-1}(a) \cap L(x)$ for each $x \in F^{-1}(b)$.

(*) Since $F^{-1}(a)$ is compact and the partial order on X is closed, g is continuous.

Define a function $k : X \times [0, 1] \to X$ by letting

$$k(x, a) \in L(x) \cap F^{-1}\big(\min\{a, F(x)\}\big)$$

for each $(x, a) \in X \times [0,1]$. Then k is clearly a contraction of X onto $\{\theta\}$.

If $a \in]0, 1[$, then $F^{-1}(a)$ is the common boundary of its complementary domains $F^{-1}([0, a[)$ and $F^{-1}(]a, 1])$. Since $F^{-1}(a)$ is nowhere dense, $F^{-1}(a)$ is not 3-dimensional. Since $F^{-1}(a)$ is the image of a 2-sphere under the map g, $F^{-1}(a)$ is a continuum. It is clear that $F^{-1}(a)$ is non-degenerate. If $y \in F^{-1}([0, a])$, then the restriction of the map k to

$$\big(F^{-1}([0, a[) \cup \{y\}\big) \times [0, 1]$$

is a contraction of $F^{-1}([0, a[) \cup \{y\}$ in $F^{-1}([0, a[) \cup \{y\}$ onto $\{\theta\}$. Thus $F^{-1}(a)$ is r-accessible from $F^{-1}([0, a[)$ for $r = 0, 1$ and 2. By Theorem V. 19. 5 in Wilder [12], $F^{-1}([0, a[)$ is semi-1-connected. By Theorem XII. 3. 9 in Wilder [12], $F^{-1}(a)$ is a 2-sphere.

We must prove that F is a 0-regular map. Since F is a homeomorphism when restricted to $\{s\} \times [0, 1]$, where $s \in \mathrm{Max}(X)$, F is an open map.

Let $p \in F^{-1}(a)$ for some $a \in]0, 1]$ and let U be an open neighbourhood of p. To prove that F is 0-regular it will suffice to prove that there exists a compact neighbourhood N of p with $N \subset U$ such that if $x \in [0, 1]$ and $F^{-1}(x) \cap N$ is non-void, then $F^{-1}(x) \cap N$ is connected.

Since $M(p) \cap \mathrm{Max}(X)$ is a compact set in the 2-sphere $\mathrm{Max}(X)$, there exist connected open subsets $V_1, \ldots, V_n$ of $\mathrm{Max}(X)$ such that

$$M(p) \cap \mathrm{Max}(X) \subset V_1 \cup \ldots \cup V_n.$$

For each $i = 1, \ldots, n$ let W_i denote the closure of V_i. Since the partial order on X is closed, we may pick $V_1, \ldots, V_n$, so that $L(W_i) \cap F^{-1}(a) \subset U$ for each i. We may also suppose that, for each i, $V_i \cap M(p)$ is non-void and the sets V_i are pairwise disjoint.

If $n = 1$, then $V = W_1$ is a continuum which contains $M(p) \cap F^{-1}(1)$ and $L(V) \cap F^{-1}(a) \subset U$. Suppose that $n > 1$. Then $a < 1$. Let $x_1 \in V_1 \cap M(p)$ and let $x_2 \in V_2 \cap M(p)$. For each $i = 1, 2$ let $y_{i,j}$ be a sequence in $L(x_i) \cap (M(p) \setminus \{p\})$ which converges to p. By Theorem X. 3. 2 in Wilder [12], $F^{-1}(]a, 1])$ is 0-ulc since it is a complementary domain of the 2-sphere $F^{-1}(a)$. Hence for some sufficiently large j there exists an arc A in $F^{-1}(]a, 1])$ with endpoints $y_{1,j}$ and $y_{2,j}$ such that $L(A) \cap F^{-1}(a) \subset U$. Since A is compact and $F(A) \subset]a, 1]$, there exists $b \in]a, 1]$ such that

$A \subset F^{-1}([b, 1])$. By (*), $L(A) \cap F^{-1}(b)$ is a continuum which meets both $L(V_1) \cap F^{-1}(b)$ and $L(V_2) \cap F^{-1}(b)$. By induction there exists $c \in]a, 1]$ and a continuum V in $F^{-1}(c)$ such that $L(V) \cap F^{-1}(a) \subset U$ and

$$\big(L(W_1) \cup \ldots \cup L(W_n)\big) \cap F^{-1}(c) \subset V.$$

Let p_i be a sequence in X which converges to p. Eventually,

$$M(p_i) \cap \operatorname{Max}(X) \subset V_1 \cup \ldots \cup V_n$$

since $V_1 \cup \ldots \cup V_n$ is a neighbourhood (in $\operatorname{Max}(X)$) of $M(p) \cap \operatorname{Max}(X)$ and the partial order on X is closed. For each i let $q_i \in M(p_i) \cap \operatorname{Max}(X)$ and let $r_i \in L(q_i) \cap V$. Eventually, $p_i \in L(r_i) \cup M(r_i)$ since $L(q_i)$ is an order arc. If eventually $p_i \in M(r_i)$, then $p \in M(V)$ and $a = 1$. Thus, p_i is eventually in $M(V) = V \subset L(V)$. If $p_i \in L(r_i)$, then $p_i \in L(V)$. Hence $L(V)$ is a neighbourhood of p.

Let a_i be a sequence in $[0, 1]$ which converges to a. For each i let $x_i \in L(V) \cap F^{-1}(a_i)$ and let x be a cluster point of the sequence x_i. Then $x \in L(V)$ since V is compact and the partial order on X is closed. Since F is continuous, $x \in F^{-1}(a)$. It follows from the argument above and the fact that the closed set $L(V) \cap F^{-1}(a)$ is contained in the open set U that there exist $a_1, a_2 \in [0, 1]$ such that

$$N = L(V) \cap F^{-1}([a_1, a_2])$$

is a neighbourhood of p which is contained in U. If $x \in [0, 1]$ such that $N \cap F^{-1}(x)$ is non-void, then $N \cap F^{-1}(x)$ is a continuum by (*).

By Theorem 2 there exists for each $\partial \in]0, 1]$ a homeomorphism h of $X \setminus F^{-1}([0, \partial[\,)$ onto $S^2 \times [\partial, 1]$ such that the diagram

$$\begin{array}{ccc} X \setminus F^{-1}([0, \partial[\,) & \xrightarrow{h} & S^2 \times [\partial, 1] \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & [\partial, 1] & \end{array}$$

commutes.

Let Q be an open 3-cell neighbourhood of θ such that the closure of Q is a closed 3-cell. There exists $\varepsilon > 0$ such that $F^{-1}([0, \varepsilon]) \subset Q$. By the argument above, $F^{-1}(\varepsilon)$ is a bicollared 2-sphere (see Brown [2]) in Q. By a theorem of Brown [2], $F^{-1}(\varepsilon)$ is flat and so $F^{-1}([0, \varepsilon])$ is a closed 3-cell. Hence

$$X = F^{-1}([0, \varepsilon]) \cup F^{-1}([\varepsilon, 1])$$

is a closed 3-cell.

A *crumpled cube* is the closure of the bounded complementary domain of a 2-sphere in E^3. The proof of Theorem 3 can be used to prove the following

THEOREM 4. *Let X be a crumpled cube with boundary S^2 and let $\theta \in X \setminus S^2$. Then X is a 3-cell if and only if X admits a closed partial order* *such that* $\mathrm{Min}(X) = \{\theta\}$, $\mathrm{Max}(X) = S^2$ *and, for each* $x \in X$, $L(x)$ *is an order arc.*

By placing strong conditions on the partial order, we can drop the hypothesis in Theorem 3 that X is a manifold.

If X is a compact metric space, we let 2^X denote the space of compact subsets of X with the Hausdorff metric.

Theorem 5. *Let X be a partially ordered space. If X contains a family $\mathscr{C}$ of maximal chains of X such that*

(i) *each member of $\mathscr{C}$ is a non-degenerate order arc,*

(ii) *$\mathscr{C}$ is homeomorphic to the closed unit disk in the plane, and*

(iii) *for each $x \in X$, $\{C \in \mathscr{C} \mid x \in C\}$ is a continuum which does not separate $\mathscr{C}$ and which does not contain the boundary of $\mathscr{C}$,*

then X is a 3-cell.

Proof. By [10], $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$ are compact since $\mathscr{C}$ is a compact family of maximal chains which covers X.

By the proof of Theorem 3 there exists a continuous function F from X onto $[0, 1]$ such that $F^{-1}(0) = \mathrm{Min}(X)$, $F^{-1}(1) = \mathrm{Max}(X)$ and, for each $a \in [0, 1]$, $F^{-1}(a)$ meets each member of $\mathscr{C}$ in precisely one point.

Define $g : \mathscr{C} \times [0, 1] \to X$ by letting

$$g(C, a) \in C \cap F^{-1}(a)$$

for each $(C, a) \in \mathscr{C} \times [0, 1]$. Then g is a continuous mapping of the 3-cell $\mathscr{C} \times [0, 1]$ onto the Hausdorff space X. For each $x \in X$, $g^{-1}(x)$ lies in the 2-cell $\mathscr{C} \times \{F(x)\}$ of $\mathscr{C} \times [0, 1]$ and

$$g^{-1}(x) = \{(C, F(x)) \mid x \in C\}.$$

Hence $g^{-1}(x)$ is a continuum which does not separate $\mathscr{C} \times \{F(x)\}$ and which does not contain the boundary of $\mathscr{C} \times \{F(x)\}$. It follows by a theorem of R. L. Moore (Whyburn [11], p. 173) that $F^{-1}(a) = g(\mathscr{C} \times \{a\})$ is a closed 2-cell for each $a \in [0, 1]$. By the proof of Theorem 8 in [5], F is 0-regular. Hence X is a 3-cell by Theorem 2.

THEOREM 6. *Let X be a partially ordered space such that*

(i) $\mathrm{Max}(X)$ *is a closed 2-cell and* $\mathrm{Min}(X)$ *is compact,*

(ii) *for each $x \in X$, $L(x)$ is a non-degenerate order arc, and*

(iii) *for each $x \in X$, $M(x) \cap \mathrm{Max}(X)$ is a continuum which does not separate* $\mathrm{Max}(X)$ *and which does not contain the boundary of* $\mathrm{Max}(X)$.

Then X is a 3-cell.

Proof. By Theorem 5 we need only to show that the function $f: \mathrm{Max}(X) \to 2^X$, defined by letting $f(x) = L(x)$ for each $x \in \mathrm{Max}(X)$, is continuous.

Let x_i be a sequence in $\mathrm{Max}(X)$ which converges to x in $\mathrm{Max}(X)$. Since 2^X is compact, the sequence $L(x_i)$ has a cluster point A in 2^X. Since the partial order on X is closed, $A \subset L(x)$. For each i, $L(x_i)$ meets $\mathrm{Min}(X)$. Since $\mathrm{Min}(X)$ is closed, A meets $\mathrm{Min}(X)$. Finally, since each $L(x_i)$ is connected, A is connected. Now, A is a connected set in $L(x)$ which meets both $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$. Since $L(x)$ is an irreducible arc between $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$, $A = L(x)$. Hence f is continuous.

3. Convex metrics. We can restate the results of Section 2 as theorems about spaces which admit certain kinds of metrics.

By Carruth's theorem a partially ordered space X admits a metric ϱ such that every chain in X is isometric to a chain in $[0, 1]$. The following result provides a converse to this theorem.

PROPOSITION 7. *Let ϱ be a metric for the compact, metric space X and let $\theta \in X$. Define a relation $\leqslant$ on $X \times X$ be letting $(x, y) \in \leqslant$ if and only if $x = y$ or $\varrho(x, \theta) < \varrho(y, \theta)$ and $\varrho(\theta, x) + \varrho(x, y) = (\theta, y$. Then $\leqslant$ is a closed partial order on X.*

The proof is straightforward.

Let ϱ be a metric for the compact metric space X. An arc A in X will be called a *line segment* if A is isometric to an arc in $[0, 1]$.

Let Y be a closed subset of X and let $\theta \in Y \setminus X$. The metric ϱ will be called *θ-Y convex* if for each $x \in X$ there is a line segment which contains θ and x and if each line segment which is maximal among the line segments with endpoint θ meets Y in precisely one point. The θ-Y convex metric ϱ will be called *strongly θ-Y convex* if for each $y \in Y$ there is exactly one line segment from θ to y.

PROPOSITION 8. *Let Y be a closed subset of the compact metric space X and let $\theta \in X \setminus Y$. Then X admits a θ-Y convex metric ϱ if and only if X admits a closed partial order such that* $\mathrm{Min}(X) = \{\theta\}$, $\mathrm{Max}(X) = Y$ *and, for each $x \in X$, $L(x) \cup M(x)$ is connected.*

Proof ($\Rightarrow$). Suppose ϱ is a θ-Y convex metric for X. Let $\leqslant$ be the partial order defined in Proposition 7. Let $x \leqslant^* y$ if $x \leqslant y$ and x and y lie in a line segment from θ to Y. Then $\leqslant^*$ is the required partial order.

($\Leftarrow$) Suppose X admits a closed partial order such that $\mathrm{Min}(X) = \{\theta\}$, $\mathrm{Max}(X) = Y$ and, for each $x \in X$, $L(x) \cup M(x)$ is connected. By a theorem of R. J. Koch (see Ward [9]), each point of X lies in an order arc which runs from $\mathrm{Min}(X)$ to $\mathrm{Max}(X)$. One can check that if ϱ is a radially convex metric for X which is defined by the proof of Carruth's theorem and x, y and z are distinct points in X, then

$$\varrho(x, z) = \varrho(x, y) + \varrho(y, z)$$

if and only if either $x < y < z$ or $z < y < x$. Then ϱ is a θ-Y convex metric for X.

From Proposition 8 and Theorem 3 we get

THEOREM 9. *Let X be a compact 3-manifold with boundary S^2. Then X is a 3-cell if and only if X admits a strongly θ-S^2 convex metric.*

THEOREM 10. *If X is a compact n-manifold with connected boundary B, then X admits a θ-B convex metric for some $\theta \in X \setminus B$.*

Proof. By [3] there exists a map φ from $B \times [0, 1]$ onto X such that $\varphi(b, 1) = b$ for each $b \in B$, restriction $\varphi|_{B\times]0,1]}$ is a homeomorphism and $\varphi^{-1}(\varphi(B \times \{0\})) = B \times \{0\}$.

By [1], there exists a convex metric ϱ (i. e. every pair of points is joined by a line segment) for the Peano continuum $\varphi(B \times \{0\})$. Let $\theta \in \varphi(B \times \{0\})$ and set $x \leqslant' y$ in $\varphi(B \times \{0\})$ if and only if $x = y$ or $\varrho(\theta, x) < \varrho(\theta, y)$ and x lies on a line segment with endpoints θ and y. Then $<'$ is a closed partial order on $\varphi(B \times \{0\})$.

Define $\leqslant^*$ on $X \times X$ by letting $\varphi(r, a) \leqslant^* \varphi(s, b)$ if $a \leqslant b$ and $r = s$ or if $a = 0$ and $\varphi(r, 0) \leqslant' \varphi(s, 0)$. Then $\leqslant^*$ is a closed partial order for X such that $\mathrm{Min}(X) = \{\theta\}$, $\mathrm{Max}(X) = B$ and, for each $x \in X$, $L(x) \cup M(x)$ is connected. The theorem follows by Proposition 8.

Remarks. 1. Let X be a homotopy 3-cell with boundary S^2. If the map φ in the proof of Theorem 10 is monotone, then $\varphi(S^2 \times \{0\})$ is a tree and the metric on X given by Theorem 10 is strongly θ-S^2 convex. Hence X is a 3-cell by Proposition 9.

2. Let X be a homotopy 3-cell with boundary S^2. The map φ in the proof of Theorem 10 can be taken to be a piecewise linear map. In this case $T = \varphi(S^2 \times \{0\})$ will be a contractible 2-complex. One can prove by the methods used in this paper that T admits a closed partial order with a unique minimal element such that $L(t)$ is for each $t \in T$ an order arc if and only if T is collapsible. Thus, the method of Theorem 10 is of little use in getting a *strongly* θ-S^2 convex metric for X. It is not difficult to construct a closed partial order on X such that

(i) $\mathrm{Min}(X) = \{\theta\}$ and $\mathrm{Max}(X) = S^2$,

(ii) for each $x \in X$, $L(x)$ is connected,

(iii) there exist $x_1, \ldots, x_n \in T$ such that if $x \neq x_i$, then $L(x) \cap U$ is an order arc for some neighbourhood U of x.

Let $F: X \to [0, 1]$ be defined as in Theorem 3. If $a \in]0, 1[$ and $a \neq F(x_i)$, then $F^{-1}(a)$ is an orientable 2-manifold and $F^{-1}(F(x_i))$ may be taken to be a 2-manifold with precisely one singular point.

4. A product theorem. We turn our attention to more general spaces.

THEOREM 11. *Let X be a compact metric partially ordered space such that* $\mathrm{Min}(X)$ *and* $\mathrm{Max}(X)$ *are closed. If, for each $x \in X$, $L(x) \cup M(x)$ is a non-degenerate order arc, then X is homeomorphic to* $\mathrm{Max}(X) \times [0, 1]$.

Proof. By the proof of Theorem 3 there exists a continuous function F from X onto $[0, 1]$ such that $F^{-1}(0) = \mathrm{Min}(X)$, $F^{-1}(1) = \mathrm{Max}(X)$

and, for each $a \in [0, 1]$, $F^{-1}(a)$ meets $L(m)$ in precisely one point for each $m \in \mathrm{Max}(X)$.

Define $g: \mathrm{Max}(X) \times [0, 1] \to X$ by letting $g(m, a) \in L(m) \cap F^{-1}(a)$ for each $(m, a) \in \mathrm{Max}(X) \times [0, 1]$. Then g is clearly a one-one function onto X. Since the partial order on X is closed and F is continuous, it follows that g is continuous. Thus, g is the required homeomorphism.

THEOREM 12. *Let M be a compact metric space. If f is a continuous function from $M \times [0, 1]$ onto a Hausdorff space Y such that, for each $m \in M$, $f|_{\{m\} \times [0,1]}$ is a non-trivial monotone map and $f^{-1}(f(\{m\} \times [0, 1])) = \{m\} \times [0, 1]$, then Y is homeomorphic to $M \times [0, 1]$.*

Proof. Set $x \leqslant^* y$ in Y if and only if there exist $(m, a) \in f^{-1}(x)$ and $(m, b) \in f^{-1}(y)$ with $a \leqslant b$ in $[0, 1]$. Then $\leqslant^*$ is a closed partial order on Y and Y with this partial order satisfies the hypothesis of Theorem 11.

5. A fixed point theorem. We say that a set is *acyclic* if it has the homology of a point.

We shall need the following theorem from [6]:

THEOREM 13 (Eilenberg and Montgomery). *Let M be an acyclic absolute neighbourhood retract and let $T: M \to M$ be an upper semi-continuous multi-valued function such that, for each $x \in M$, $T(x)$ is acyclic. Then T has a fixed point.*

We can now prove our final result:

THEOREM 14. *Let X be a compact metric partially ordered space such that*

(i) $\mathrm{Max}(X)$ *is an absolute retract and* $\mathrm{Min}(X)$ *is compact,*

(ii) *for each $x \in \mathrm{Max}(X)$, $L(x)$ is a non-degenerate order arc, and*

(iii) *for each $x \in X$, $\mathrm{Max}(X) \cap M(x)$ is acyclic.*

Then X has the fixed point property.

Proof. By the proof of Theorem 5 there exists a map

$$g: \mathrm{Max}(X) \times [0, 1] \to X$$

such that, for each $m \in \mathrm{Max}(X)$, $g(m, 1) = m$ and g maps $\{m\} \times [0, 1]$ homeomorphically onto $L(m)$. For each $x \in X$, $g^{-1}(x)$ is homeomorphic to $\mathrm{Max}(X) \cap M(x)$.

Let $f: X \to X$ be a map. Then $g^{-1} \circ f \circ g$ is an upper semi-continuous multi-valued function of the absolute retract $\mathrm{Max}(X) \times [0, 1]$ into itself. For each $(T, a) \in \mathrm{Max}(X) \times [0, 1]$, $g^{-1} \circ f \circ g(T, a)$ is homeomorphic to $\mathrm{Max}(X) \cap M(f \circ g(T, a))$. Hence, the point images of $g^{-1} \circ f \circ g$ are acyclic and $g^{-1} \circ f \circ g$ has a fixed point (S, b), by Theorem 13. Thus, $(S, b) \in g^{-1} \circ f \circ g(S, b)$ and $g(S, b) = f \circ g(S, b)$. Hence $g(S, b)$ is a fixed point of f, q. e. d.

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