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GENERALIZATIONS OF CASTAING'S THEOREM ON SELECTORS

BY

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1. Introduction. Let T be a non-empty set and $\mathscr{I}$ a family of subsets of T. A function $F\colon T \to 2^X$, where 2^X denotes the space of non-empty closed subsets of a metric space X, is said to be *weakly $\mathscr{I}$-measurable* (*$\mathscr{I}$-measurable*) if $\{t\colon F(t) \cap V \neq \emptyset\} \in \mathscr{I}$ for every open (closed) set V in X. If Y is a metric space, we say that a function $f\colon T \to Y$ is *$\mathscr{I}$-measurable* if $f^{-1}(V) \in \mathscr{I}$ for every open set V in Y. A function $f\colon T \to X$ is said to be a *selector* for $F\colon T \to 2^X$ if $f(t) \in F(t)$ for each $t \in T$.

In [1], Castaing essentially established the following

THEOREM 1. *Let $(T, \mathscr{I})$ be a measurable space. Suppose $F\colon T \to 2^X$, where X is a Polish space. Then the following conditions are equivalent*:

(i) *F is weakly $\mathscr{I}$-measurable,*

(ii) *there exist $\mathscr{I}$-measurable functions $g_i\colon T \to X$, $i \geqslant 1$, such that $F(t) = \mathrm{cl}\big(\{g_i(t)\colon i \geqslant 1\}\big)$ for each $t \in T$, where* cl *denotes the closure operator.*

Actually, Castaing proved Theorem 1 under the additional assumptions that T is a locally compact space and $\mathscr{I}$ is the σ-field of μ-measurable sets for some Radon measure μ, and Himmelberg [2] observed that the result holds for any measurable space $(T, \mathscr{I})$, which is just the version we have formulated in Theorem 1.

The present article arose from our efforts to formulate Theorem 1 in the same framework as the one in which the selection theorem of Kuratowski and Ryll-Nardzewski [5] is formulated, or the even more general framework of [6]. Indeed, the present article can be viewed as a sequel to [6].

Our main results are reported in Section 2. Section 3 contains an application of Theorem 1. We conclude with an example in Section 4.

2. Main results. We first recall some definitions and notation from [6].

In what follows we identify cardinals with initial ordinals. α, β with or without primes denote ordinals, while λ denotes an infinite cardinal. λ^+ denotes the successor cardinal to λ.

Let $\mathscr{I}$ be a family of subsets of a set T. We say that $\mathscr{I}$ is *λ-additive* (*λ-multiplicative*) if, whenever $\{A_\alpha: \alpha < \beta\} \subseteq \mathscr{I}$ and $\beta < \lambda$,

$$\bigcup_{\alpha<\beta} A_\alpha \in \mathscr{I} \quad \Big(\bigcap_{\alpha<\beta} A_\alpha \in \mathscr{I}\Big).$$

Denote by $\mathscr{I}_\lambda$ the smallest λ-additive family containing $\mathscr{I}$. We sometimes write $\mathscr{I}_\sigma$ for $\mathscr{I}_{\aleph_1}$. We denote by $\mathscr{I}^c$ the family of subsets of T whose complements belong to $\mathscr{I}$. Say that $\mathscr{I}$ is a *λ-field* if $\emptyset \in \mathscr{I}$, $\mathscr{I}$ is λ-additive and closed under complementation. Thus an $\aleph_0$-field is just a field, while an $\aleph_1$-field is a σ-field.

We say that the family $\mathscr{I}$ satisfies the *λ-reduction principle* if, whenever $\{A_\alpha: \alpha < \beta\} \subseteq \mathscr{I}$ and $\beta < \lambda$, there exist sets B_α, $\alpha < \beta$, such that

(a) $(\forall \alpha < \beta)(B_\alpha \in \mathscr{I})$,

(b) $(\forall \alpha < \beta)(B_\alpha \subseteq A_\alpha)$,

(c) $(\forall \alpha, \alpha' < \beta)(\alpha \neq \alpha' \Rightarrow B_\alpha \cap B_{\alpha'} = \emptyset)$,

(d) $\bigcup_{\alpha<\beta} B_\alpha = \bigcup_{\alpha<\beta} A_\alpha$.

We shall need the following result of Kuratowski [4]:

LEMMA. *If $\mathscr{I}$ is a λ-field of subsets of T, then $\mathscr{I}_{\lambda^+}$ satisfies the λ^+-reduction principle.*

We are now in a position to state our main results.

THEOREM 2. *Let Φ be a family of subsets of a set T such that $\emptyset, T \in \Phi$, Φ is λ^+-additive and λ-multiplicative. Then the following conditions are equivalent:*

(a) *$\Phi = \mathscr{L}_{\lambda^+}$ for some λ-field $\mathscr{L}$ of subsets of T.*

(b) *If X is a complete metric space of topological weight $\leqslant \lambda$ and $F: T \to 2^X$ is weakly Φ-measurable, then there exist Φ-measurable functions $f_\alpha: T \to X$, $\alpha < \lambda$, such that*

$$(\forall t \in T)\big(F(t) = \mathrm{cl}(\{f_\alpha(t): \alpha < \lambda\})\big).$$

(c) *If $Z = \{0, 1\}$ with the discrete topology and $F: T \to 2^Z$ is weakly Φ-measurable, then there exist Φ-measurable functions $f_\alpha: T \to Z$, $\alpha < \lambda$, such that*

$$(\forall t \in T)\big(F(t) = \mathrm{cl}(\{f_\alpha(t): \alpha < \lambda\})\big).$$

Proof. (a) $\Rightarrow$ (b). Fix a metric d on X. It suffices to show that for every $\varepsilon > 0$ there exist Φ-measurable selectors $g_\alpha: T \to X$ for F such that $\{g_\alpha(t): \alpha < \lambda\}$ is an ε-net in $F(t)$ for each $t \in T$. Once this is done, we can put together the functions g_α^n, $\alpha < \lambda$, $n \geqslant 1$, to get the desired result, where g_α^n, $\alpha < \lambda$, are Φ-measurable selectors for F such that $\{g_\alpha^n(t): \alpha < \lambda\}$ is a $(1/n)$-net in $F(t)$ for each $t \in T$.

Let $\varepsilon > 0$. Since the topological weight of X is $\leqslant \lambda$, it is possible to cover X by open spheres S_α, $\alpha < \lambda$, of radius $\varepsilon/2$. Put $T_\alpha = \{t: F(t) \cap S_\alpha \neq \emptyset\}$, $\alpha < \lambda$. Note that $T_\alpha \in \Phi$.

Fix a Φ-measurable selector $h: T \to X$ for F. The existence of such a selector is ensured by Theorem 1 in [6], as Φ satisfies the λ^+-reduction principle (Lemma). Suppose that $T_\alpha \neq \emptyset$. Define $F_\alpha: T_\alpha \to 2^X$ by $F_\alpha(t) = \mathrm{cl}\big(F(t) \cap S_\alpha\big)$. Put $\Phi_\alpha = \{A \cap T_\alpha: A \in \Phi\}$. Note that $\Phi_\alpha \subseteq \Phi$. Now if V is open in X, then

$$\{t \in T_\alpha: F_\alpha(t) \cap V \neq \emptyset\} = \{t \in T: F(t) \cap V \cap S_\alpha \neq \emptyset\} \in \Phi,$$

so that F_α is weakly Φ_α-measurable on T_α. Since Φ satisfies the λ^+-reduction principle, so does Φ_α. We can therefore appeal once again to Theorem 1 in [6] to get a Φ_α-measurable selector $h_\alpha: T_\alpha \to X$ for F_α. Since $T_\alpha \in \Phi = \mathscr{L}_{\lambda^+}$, there exist sets $T_{\alpha\beta} \in \mathscr{L}$ such that

$$T_\alpha = \bigcup_{\beta<\lambda} T_{\alpha\beta}.$$

For $\beta < \lambda$, define $h_{\alpha\beta}: T \to X$ as follows:

$$h_{\alpha\beta} = \begin{cases} h_\alpha & \text{on } T_{\alpha\beta}, \\ h & \text{on } T - T_{\alpha\beta}. \end{cases}$$

If $T_\alpha = \emptyset$, set $h_{\alpha\beta} = h$ for every $\beta < \lambda$. In either case, as is easy to check, the functions $h_{\alpha\beta}$, $\beta < \lambda$, are Φ-measurable selectors for F.

To conclude the proof we claim that, for each $t \in T$, $\{h_{\alpha\beta}(t): \alpha, \beta < \lambda\}$ is an ε-net in $F(t)$. For if $x \in F(t)$, we can find an α such that $x \in S_\alpha$, hence $F(t) \cap S_\alpha \neq \emptyset$ and $t \in T_\alpha$. So there is a β such that $t \in T_{\alpha\beta}$. It follows that $h_{\alpha\beta}(t) = h_\alpha(t) \in F_\alpha(t) \subseteq \mathrm{cl}(S_\alpha)$. Consequently, $d\big(x, h_{\alpha\beta}(t)\big) < \varepsilon$.

(b) $\Rightarrow$ (c). This implication is obvious.

(c) $\Rightarrow$ (a). Put $\mathscr{L} = \Phi \cap \Phi^c$. It is easy to see that $\mathscr{L}$ is a λ-field of subsets of T. We now claim that $\Phi = \mathscr{L}_{\lambda^+}$. The inclusion $\mathscr{L}_{\lambda^+} \subseteq \Phi$ is obvious. For the reverse inclusion, let $A \in \Phi$ and suppose $\emptyset \neq A \neq T$. Let $Z = \{0, 1\}$ be equipped with the discrete topology. Define $F: T \to 2^Z$ as follows:

$$F(t) = \begin{cases} \{0, 1\} & \text{if } t \in A, \\ \{1\} & \text{if } t \in T - A. \end{cases}$$

It is easy to check that F is weakly Φ-measurable. It now follows, by virtue of condition (c), that there exist Φ-measurable functions $f_\alpha: T \to X$, $\alpha < \lambda$, such that

$$F(t) = \{f_\alpha(t): \alpha < \lambda\} \quad \text{for each } t \in T.$$

Now verify that $f_\alpha^{-1}(\{0\}) \in \Phi \cap \Phi^c = \mathscr{L}$ for each $\alpha < \lambda$ and

$$A = \bigcup_{\alpha<\lambda} f_\alpha^{-1}(\{0\}),$$

so that $A \in \mathscr{L}_{\lambda^+}$.

This completes the proof of Theorem 2.

THEOREM 3. *Let $\mathscr{L}$ be a λ-field of subsets of a set T and let X be a complete metric space of topological weight $\leqslant \lambda$. Suppose $F\colon T \to 2^X$. Then the following conditions are equivalent:*

(i) *F is weakly $\mathscr{L}_{\lambda^+}$-measurable,*

(ii) *there exist $\mathscr{L}_{\lambda^+}$-measurable functions $f_a\colon T \to X$, $a < \lambda$, such that*

$$(\forall t \in T)\big(F(t) = \mathrm{cl}(\{f_a(t)\colon a < \lambda\})\big).$$

Proof. (i) $\Rightarrow$ (ii) is just the implication (a) $\Rightarrow$ (b) of Theorem 2. For (ii) $\Rightarrow$ (i), let V be an open set in X. Then

$$\{t\colon F(t)\cap V \neq \emptyset\} = \bigcup_{a<\lambda} f_a^{-1}(V) \in \mathscr{L}_{\lambda^+},$$

so that F is weakly $\mathscr{L}_{\lambda^+}$-measurable.

A generalization of the selection theorem of Kuratowski and Ryll-Nardzewski [5] is obtained by specializing Theorem 3 to the case $\lambda = \aleph_0$. By further specializing Theorem 3 to the case where $\mathscr{L}$ is a σ-field, we recover (Castaing's) Theorem 1. Furthermore, in the case $\lambda = \aleph_0$, Theorem 2 shows that Castaing type theorems do not hold for structures on T more general than $\mathscr{L}_\sigma$, where $\mathscr{L}$ is a field, unless further conditions are imposed on T.

Theorem 3 yields a characterization of multifunctions of class a^- as follows. Let T and X be metric spaces. Recall that a function $F\colon T \to 2^X$ is said to be of *class a^-* if $\{t\colon F(t)\cap V \neq \emptyset\}$ is a Borel set in T of additive class a for each open set V in X. We say that a function $f\colon T \to X$ is a *Borel function of class a* if $f^{-1}(V)$ is a Borel set in T of additive class a for each open set V in X.

THEOREM 4. *Let T be a metric space and let X be a Polish space. Suppose $F\colon T \to 2^X$ and $a > 0$. Then the following conditions are equivalent:*

(i) *F is of class a^-,*

(ii) *there exist functions $f_i\colon T \to X$, $i \geqslant 1$, such that f_i is a Borel function of class a and*

$$(\forall t \in T)\big(F(t) = \mathrm{cl}(\{f_i(t)\colon i \geqslant 1\})\big).$$

Proof. The result is an immediate consequence of Theorem 3 by taking $\lambda = \aleph_0$ and $\mathscr{L}$ to be the family of Borel subsets of T which are simultaneously of additive class a and multiplicative class a.

Theorem 4 is also true for $a = 0$ if we assume that T is a 0-dimensional separable metric space.

3. An application. In this section we use Castaing's theorem to give a quick proof of a result which is useful in dynamic programming and control theory. Variants of the result abound in the literature. The reader is referred to the survey article [7] for a bibliography.

We fix some notation first. We use $\mathscr{C}(X)$ to denote the space of non-empty compact subsets of a metric space X. If $F\colon T \to \mathscr{C}(X)$, we denote the set $\{(t, x)\colon x \in F(t)\}$ by $\mathrm{Gr}F$. The Borel σ-field of the metric space X is denoted by $\mathscr{B}_X$. If $(T, \mathscr{S})$ is a measurable space and X a metric space, $\mathscr{S} \otimes \mathscr{B}_X$ is the product of the σ-fields $\mathscr{S}$ and $\mathscr{B}_X$. Finally, if $E \subseteq T \times X$, $\mathscr{S} \otimes \mathscr{B}_X | E$ denotes the trace of the σ-field $\mathscr{S} \otimes \mathscr{B}_X$ on E.

THEOREM 5. *Suppose $(T, \mathscr{S})$ is a measurable space, $F\colon T \to \mathscr{C}(X)$ is weakly measurable (equivalently, measurable), where X is a separable metric space. Let $u\colon \mathrm{Gr}F \to (-\infty, \infty)$ be $\mathscr{S} \otimes \mathscr{B}_X | \mathrm{Gr}\,F$-measurable and assume moreover that $u(t, \cdot)$ is continuous on $F(t)$ for each $t \in T$. Let $v\colon T \to (-\infty, \infty)$ be defined by*

$$v(t) = \sup_{x \in F(t)} u(t, x).$$

Then v is $\mathscr{S}$-measurable and there is an $\mathscr{S}$-measurable selector $f\colon T \to X$ for F such that $v(t) = u\big(t, f(t)\big)$ for each $t \in T$.

Proof. A moment's reflection shows that we may consider X to be a Polish space (or even a compact metric space) without loss of generality. With this assumption made, we can invoke (Castaing's) Theorem 1 to get $\mathscr{S}$-measurable functions $g_i\colon T \to X$, $i \geqslant 1$, such that $F(t) = \mathrm{cl}\big(\{g_i(t)\colon i \geqslant 1\}\big)$ for each $t \in T$. It follows immediately that

$$v(t) = \sup_i u\big(t, g_i(t)\big) \quad \text{for each } t \in T.$$

Hence v is $\mathscr{S}$-measurable.

Set $G(t) = \{x \in F(t)\colon u(t, x) \geqslant v(t)\}$, $t \in T$. Plainly, $G\colon T \to \mathscr{C}(X)$. We show next that G is $\mathscr{S}$-measurable. Let then C be a closed set in X and let

$$C_n = \{x \in X\colon d(x, C) < 1/n\}, \quad n \geqslant 1,$$

where d is a metric on X. Note that

$$C = \bigcap_{n \geqslant 1} C_n.$$

Check now that

$$\{t\colon G(t) \cap C \neq \emptyset\} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \big\{t\colon u\big(t, g_i(t)\big) > v(t) - 1/n \text{ and } g_i(t) \in C_n\big\}.$$

It follows that G is $\mathscr{S}$-measurable.

To complete the proof of Theorem 5, it suffices now to get an $\mathscr{S}$-measurable selector $f\colon T \to X$ for G.

4. Example. Himmelberg and Van Vleck ([3], Theorem 1′ (i)) and also Wagner ([7], Theorem 4.2 (e)) assert that in Theorem 1 the condition that F be weakly $\mathscr{S}$-measurable can be replaced by the following one:

(*) $\{t\colon F(t) \cap K \neq \emptyset\} \in \mathscr{S}$ for each compact subset K of X.

Here is an example showing this to be false.

Let $T = X$ be the space of irrationals. Let $\mathscr{I}$ be the σ-field on T generated by the compact subsets of T. Define F: $T \to \mathscr{C}(X)$ by $F(t) = \{t\}$. It is easy to see that F satisfies condition $(*)$. Now the only selector for F is the function $f(t) = t$, $t \in T$. But, as is well known, f is not $\mathscr{I}$-measurable. In other words, F does not admit an $\mathscr{I}$-measurable selector.

REFERENCES

[1] C. Castaing, *Sur les multi-applications mesurables*, Revue Française d'Informatique et de Recherche Opérationnelle 1 (1967), p. 91-126.

[2] C. J. Himmelberg, *Measurable relations*, Fundamenta Mathematicae 87 (1975), p. 53-72.

[3] — and F. S. Van Vleck, *Multifunctions on abstract measurable spaces and application to stochastic decision theory*, Annali di Matematica Pura ed Applicata 101 (1974), p. 229-236.

[4] K. Kuratowski, *Sur les théorèmes de séparation dans la théorie des ensembles*, Fundamenta Mathematicae 26 (1936), p. 183-191.

[5] — and C. Ryll-Nardzewski, *A general theorem on selectors*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 13 (1965), p. 397-403.

[6] A. Maitra and B. V. Rao, *Selection theorems and the reduction principle*, Transactions of the American Mathematical Society 202 (1975), p. 57-66.

[7] D. H. Wagner, *Survey of measurable selection theorems*, SIAM Journal on Control and Optimization 15 (1977), p. 859-903.

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