COLLOQUIUM MATHEMATICUM

VOL. XXXVII 1977 FASC. 2

CONTINUA WITH A CONNECTED SET OF POINTS OF INDECOMPOSABILITY

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In [2] Proffitt introduced and studied the set E_M of points of indecomposability for an arbitrary continuum M: a point p *belongs to* E_M if there exists no decomposition of M into proper subcontinua K and L such that $p \in K \cap L$. The aim of this paper is to study how the continuum M is built if E_M is a proper and connected subset of M. We show that, in this case, M is a union of two proper subcontinua K and L such that K contains E_M, L is indecomposable and disjoint with E_M, and each composant of L intersects $K \cap L$. Thus, identifying K to a single point we get the quotient space M/K which is an indecomposable continuum having precisely one composant. The existence of a Hausdorff continuum M for which E_M is a proper and connected subset of M is an open question.

1. Preliminaries. Let us note some facts which will be used in the sequel.

1.1. THEOREM (Proffitt [2]). *If T is a component of E_M, then either T is closed or* $\mathrm{cl}\, T$ *is indecomposable.*

1.2. COROLLARY. *If E_M is connected, then E_M is closed or* $\mathrm{cl}\, E_M$ *is indecomposable.*

1.3. THEOREM (Proffitt [2]). *If M is a continuum such that E_M has two components, then E_M has only two components and no proper subcontinuum of M intersects both of them.*

1.4. THEOREM (Kuratowski [1], Theorem 1, p. 131). *Let C be a connected subset of a connected space X. If A and B are two separated sets such that $X \setminus C = A \cup B$, then the sets $C \cup A$ and $C \cup B$ are connected. If, in addition, C is closed, then so are $C \cup A$ and $C \cup B$.*

2. Continua M for which E_M are non-empty and connected. Throughout this section, E_M will be a non-empty, connected, proper subset of a continuum M.

2.1. LEMMA. *If K and L are proper subcontinua such that*

$$K \cup L = M \quad \text{and} \quad K \cap E_M \neq \emptyset,$$

then $L \cap E_M = \emptyset$.

Proof. K and L are proper subcontinua of M, hence $K \cap E_M$ and $L \cap E_M$ are closed subsets of E_M such that

$$(K \cap E_M) \cup (L \cap E_M) = E_M \quad \text{and} \quad (K \cap E_M) \cap (L \cap E_M) = K \cap L \cap E_M = \emptyset.$$

Since E_M is connected and $K \cap E_M \neq \emptyset$, we have $L \cap E_M = \emptyset$.

2.2. LEMMA. *If K is a subcontinuum of M such that*

$$K \cap E_M \neq \emptyset \quad \text{and} \quad K \setminus E_M \neq \emptyset,$$

then there exists a subcontinuum L of M such that

$$L \cap E_M = \emptyset \quad \text{and} \quad K \cup L = M.$$

Proof. Let $p \in K \setminus E_M$. There exist proper subcontinua K' and L such that $K' \cup L = M$ and $p \in K' \cap L$. It follows from 2.1 that

$$K' \cap E_M = \emptyset \quad \text{or} \quad L \cap E_M = \emptyset.$$

Assume that $L \cap E_M = \emptyset$. We prove that $K \cup L = M$. In fact, $K \cup L$ is a continuum such that $p \in K \cap L$, and the assumption $K \cup L \neq M$ implies that

$$(K \cup L) \cup K' = M \quad \text{and} \quad E_M \cap (K \cup L) \cap K' \neq \emptyset$$

(since $E_M \subset K'$), a contradiction. Hence $K \cup L = M$.

2.3. COROLLARY. *If $K \cap E_M \neq \emptyset$, then $E_M \subset K$ or $K \subset E_M$.*

2.4. LEMMA. *E_M is a subset of M with a void interior.*

Proof. Suppose, on the contrary, that $\operatorname{int} E_M \neq \emptyset$. Take an open subset U of M satisfying $\operatorname{cl} U \subset \operatorname{int} E_M$. Since E_M is a proper subset of M, there are two proper subcontinua K and L of M such that

$$K \cup L = M \quad \text{and} \quad K \cap L \neq \emptyset.$$

Assume that $K \cap E_M \neq \emptyset$. By Lemma 2.1, $L \cap E_M = \emptyset$, so that $E_M \subset K$. The component C of L in $M \setminus U$ is a proper subcontinuum of M which intersects $\operatorname{cl} U$. Therefore, we have two proper subcontinua C and K of M such that

$$M = C \cup K \quad \text{and} \quad \emptyset \neq \operatorname{cl} U \cap C \cap K \subset E_M,$$

which is a contradiction.

2.5. THEOREM. *E_M is a closed subset of M iff E_M is nowhere dense; otherwise, $\mathrm{cl}\,E_M$ is indecomposable.*

Proof. By Corollary 1.2 it suffices to prove that if E_M is nowhere dense, then E_M is closed. Suppose, on the contrary, that there exists a $p \in \mathrm{cl}\,E_M \setminus E_M$. Take proper subcontinua K and L such that $K \cup L = M$ and $p \in K \cap L$. Assume that $L \cap E_M = \emptyset$ (by Lemma 2.1). The set E_M is nowhere dense and $p \in L \cap \mathrm{cl}\,E_M$, hence $L \cup \mathrm{cl}\,E_M$ is a proper subcontinuum of M. Take, according to Lemma 2.2, a proper subcontinuum K' of M such that

$$K' \cup (\mathrm{cl}\,E_M \cup L) = M.$$

By 2.1, $K' \cap E_M = \emptyset$. But $L \cap E_M = \emptyset$, so

$$\emptyset \neq E_M \subset M \setminus (K' \cup L),$$

and $M \setminus (K' \cup L)$ is open. However, $M \setminus (K' \cup L) \subset \mathrm{cl}\,E_M$ which contradicts the fact that E_M is nowhere dense.

Define $\mathscr{K}$ to be the family of all proper subcontinua K of M with $K \cap E_M \neq \emptyset$. Define $\mathscr{L}$ to be the family of all proper subcontinua L of M which satisfy $L \cap E_M = \emptyset$ and for which there exists a $K \in \mathscr{K}$ such that $L \cup K = M$.

2.6. THEOREM. *If $L \in \mathscr{L}$, then $M \setminus L$ is connected.*

Proof. Suppose that $M \setminus L$ is not connected. Let C be the component of $M \setminus L$ containing E_M. We have

$$L \cap \mathrm{cl}\,C \neq \emptyset \quad \text{and} \quad \mathrm{cl}\,C \neq M \setminus L.$$

Therefore, $L \cup \mathrm{cl}\,C$ is a proper subcontinuum of M and contains E_M. But $L \in \mathscr{L}$, hence there exists another proper subcontinuum K of M, which contains E_M and is such that

$$M = K \cup L = K \cup (L \cup \mathrm{cl}\,C);$$

a contradiction.

2.7. THEOREM. *If $L \in \mathscr{L}$ and $L \cap \mathrm{cl}\,E_M = \emptyset$, then $N = \mathrm{cl}(M \setminus L)$ is a continuum such that E_N is a proper subset of N which contains the set $E_M \cup (L \cap N)$.*

Proof. In virtue of 2.6, N is a continuum. Obviously, N is a proper subcontinuum of M and $L \cup N = M$. Now, E_M is a proper subset of N. To see this we take an open set U in M which contains $\mathrm{cl}\,E_M$ and is such that $\mathrm{cl}\,U \subset M \setminus L$. Let C be the component of U containing E_M. Then $\mathrm{cl}\,C$ is a continuum which contains E_M and differs from E_M. Obviously, $\mathrm{cl}\,C$ is contained in $\mathrm{cl}\,U$. In virtue of 2.3, there exists an $L_1 \in \mathscr{L}$ such that

$$L_1 \cup \mathrm{cl}\,C = M \quad \text{and} \quad M \setminus L_1 \subset M \setminus L.$$

The set $M \setminus L_1$ is open in N (being open in M) and, therefore, $\mathrm{cl}(M \setminus L_1)$ is a proper subcontinuum of N, contained in $\mathrm{cl}\, U$ and satisfying

$$\mathrm{int}_N \mathrm{cl}(M \setminus L_1) \neq \emptyset .$$

So, N contains a proper subcontinuum $\mathrm{cl}\, C$ which has a non-empty interior. Hence N is decomposable and, therefore, $E_N \neq N$.

Now, let A and B be proper subcontinua of N covering N. Let $p \in A \cap B$ (i.e. p is an arbitrary point of $N \setminus E_N$). Since $L \cap N \neq \emptyset$, we have

$$A \cap L \neq \emptyset \quad \text{or} \quad B \cap L \neq \emptyset,$$

say A intersects L. Assume that $p \in E_M \cup L \cap N$. If $p \in E_M$, then $L \cup A$ and N are proper subcontinua of M covering M, and $p \in N \cap (L \cup A)$; a contradiction. If $p \in L \cap N$, then $L \cup A$ and $L \cup B$ are proper subcontinua of M. Since $E_M \subset A \cup B$, then E_M intersects A or E_M intersects B, say $E_M \cap B \neq \emptyset$. Hence $L \cup B$ and N are proper subcontinua of M such that

$$L \cup B \cup N = M \quad \text{and} \quad E_M \cap (L \cup B) \cap N \neq \emptyset .$$

Thus $p \notin E_M \cup (L \cap N)$, which means that

$$N \setminus E_N \subset N \setminus E_M \cup (L \cap N), \quad \text{i.e.} \quad E_M \cup (L \cap N) \subset E_N .$$

2.8. THEOREM. *If $K \in \mathscr{K}$ and $K \not\subset E_M$, then $M \setminus K$ is connected and $\mathrm{cl}(M \setminus K) \in \mathscr{L}$.*

Proof. It follows from Corollary 2.3 that $E_M \subset K$. If $M \setminus K$ is not connected, then by Theorem 1.4 there are two continua $K \cup A$ and $K \cup B$ such that

$$(K \cup A) \cup (K \cup B) = M \quad \text{and} \quad E_M \subset (K \cup A) \cap (K \cup B),$$

where A and B are separated subsets of $M \setminus K$, a contradiction.

2.9. THEOREM. *Let $K \in \mathscr{K}$. If $L \in \mathscr{L}$ is such that*

$$K \cup L = M, \quad \mathrm{cl}\,\mathrm{int}(K \cap L) = K \cap L \quad \textit{and} \quad \mathrm{int}(K \cap L) \neq \emptyset,$$

then the set $K \cap L$ is an irreducible continuum between $L \cap \mathrm{cl}(M \setminus L)$ and $K \cap \mathrm{cl}(M \setminus K)$.

Proof. Suppose that $K \cap L = A \cup B$, where A and B are separated sets. Since $K \cap L = K \setminus (M \setminus L)$, and $M \setminus L$, in virtue of Theorem 2.6, is connected, then $(M \setminus L) \cup A$ and $(M \setminus L) \cup B$ are connected, according to Theorem 1.4. Since $\mathrm{cl}\,\mathrm{int}(K \cap L) = K \cap L$,

$$C_1 = \mathrm{cl}[(M \setminus L) \cup A] \quad \text{and} \quad C_2 = \mathrm{cl}[(M \setminus L) \cup B]$$

are proper subcontinua of K. One of them, say C_1, must intersect the continuum $\mathrm{cl}(M \setminus K)$. This produces a contradiction: K and $C_1 \cup \mathrm{cl}(M \setminus K)$ are two proper subcontinua of M which cover M and satisfy

$$E_M \subset K \cap [C_1 \cup \mathrm{cl}(M \setminus K)].$$

Now let C be a subcontinuum of $K\cap L$ such that

$$L\cap \mathrm{cl}(M\setminus L)\subset C \quad \text{and} \quad K\cap \mathrm{cl}(M\setminus K)\subset C.$$

Then

$$D=\mathrm{cl}(M\setminus L)\cup C\cup \mathrm{cl}(M\setminus K)$$

is a subcontinuum of M such that $D\cup K=M$ and $E_M\subset D\cap K$. So we have $D=M$ and, therefore,

$$\begin{aligned}C\supset M\setminus[\mathrm{cl}(M\setminus L)\cup \mathrm{cl}(M\setminus K)] &= M\setminus \mathrm{cl}[(M\setminus L)\cup(M\setminus K)]\\ &= M\setminus \mathrm{cl}(M\setminus K\cap L)=\mathrm{int}(K\cap L).\end{aligned}$$

Hence $K\cap L=\mathrm{cl}\,\mathrm{int}(K\cap L)\subset C$, so that $C=K\cap L$.

2.10. THEOREM. *If L and L_1 are in $\mathscr{L}$, then either $(L\setminus L_1)\cup(L_1\setminus L)$ is nowhere dense or $L\subset \mathrm{int}L_1$, or $L_1\subset \mathrm{int}L$.*

If K and K_1 are in $\mathscr{K}$, $K\not\subset E_M$ and $K_1\not\subset E_M$, then either $(K\setminus K_1)\cup$ $\cup(K_1\setminus K)$ is nowhere dense or $K\subset \mathrm{int}K_1$, or $K_1\subset \mathrm{int}K$.

Proof. Suppose that $(L_1\setminus L)\cup(L\setminus L_1)$ is not a nowhere dense subset of M. Then

$$\mathrm{int}(L_1\setminus L)\neq\emptyset \quad \text{or} \quad \mathrm{int}(L\setminus L_1)\neq\emptyset,$$

say $\mathrm{int}(L_1\setminus L)\neq\emptyset$. Now we show that $L\subset \mathrm{int}L_1$. In fact, if not, then a contradiction is obtained, since then

$$L\cap \mathrm{cl}(M\setminus L_1)\neq\emptyset,$$

and $L\cup \mathrm{cl}(M\setminus L_1)$ is a proper subcontinuum of M (because $L\cup \mathrm{cl}(M\setminus L_1)$ $\subset M\setminus \mathrm{int}(L_1\setminus L))$ such that

$$\mathrm{cl}(M\setminus L)\cup L\cup \mathrm{cl}(M\setminus L_1)=M \quad \text{and} \quad E_M\subset[L\cup \mathrm{cl}(M\setminus L_1)]\cap \mathrm{cl}(M\setminus L).$$

If K and K_1 are in $\mathscr{K}$, $K\not\subset E_M$ and $K_1\not\subset E_M$, then $\mathrm{cl}(M\setminus K)$ and $\mathrm{cl}(M\setminus K_1)$ are in $\mathscr{L}$ according to 2.8. By the previous case we have the required relations for K and K_1.

Let $\mathscr{K}'$ be the family of all those elements of $\mathscr{K}$ which are not contained in E_M. We write $K\Delta K'$ iff $(K\setminus K')\cup(K'\setminus K)$ is nowhere dense. The relation Δ is an equivalence in $\mathscr{K}'$.

2.11. LEMMA. *Each equivalence class of the relation Δ can be represented by some regularly closed element from $\mathscr{K}'$.*

Proof. Take an arbitrary equivalence class of the relation Δ and an element K in it. Then, by 2.8, $\mathrm{cl}(M\setminus K)$ is in $\mathscr{L}$ and, therefore, according to 2.6, $\mathrm{cl}[M\setminus \mathrm{cl}(M\setminus K)]$ is in $\mathscr{K}'$. Obviously, $\mathrm{cl}[M\setminus \mathrm{cl}(M\setminus K)]$ is regularly closed and $K\Delta\,\mathrm{cl}[M\setminus \mathrm{cl}(M\setminus K)]$.

Now, let $\mathscr{R}$ be the family of all regularly closed elements from $\mathscr{K}'$.

2.12. LEMMA. *The set* $\bigcup\{\mathrm{int}K\colon K\in\mathfrak{R}\}$ *is a proper subset of* M.

Proof. If not, then $\bigcup\{\mathrm{int}K\colon K\in\mathfrak{R}\}$ covers M. Since M is compact, there are $K_1, K_2, \ldots, K_n$ in $\mathfrak{R}$ such that

$$\mathrm{int}K_1\cup\ldots\cup\mathrm{int}K_n = M.$$

By 2.10 there exists a $K_i\in\mathfrak{R}$ such that

$$\mathrm{int}K_1\cup\ldots\cup\mathrm{int}K_n\subset K_i,$$

a contradiction.

2.13. THEOREM. *There exists a* $K_0\in\mathfrak{R}$ *such that* $K\subset K_0$ *for each* K *in* $\mathfrak{R}$.

Proof. The set $\mathrm{cl}\bigcup\{K\colon K\in\mathfrak{R}\}$ is a subcontinuum of M (each K in $\mathfrak{R}$ contains E_M). It is a proper subcontinuum of M. If not, then, by 2.10 and 2.12, $\bigcup\{\mathrm{int}K\colon K\in\mathfrak{R}\}$ is a dense and proper subset of M. Let

$$p\notin\bigcup\{\mathrm{int}K\colon K\in\mathfrak{R}\}.$$

Since $p\notin E_M$, there exists a $K'\in\mathscr{K}'$ such that $p\in K'$. Since $\bigcup\{\mathrm{int}K\colon K\in\mathfrak{R}\}$ is dense in M, and $M\setminus K'$ is not empty and open, then there exists a K in $\mathfrak{R}$ such that

$$\mathrm{int}K\cap(M\setminus K')\neq\emptyset.$$

By 2.10, $K'\subset\mathrm{int}K$, so $p\in\mathrm{int}K$; a contradiction. Thus $\mathrm{cl}\bigcup\{K\colon K\in\mathfrak{R}\}$ is a proper subcontinuum of M. Take K_0 to be this regularly closed element of $\mathscr{K}'$ which represents the equivalence class of the elements $\mathrm{cl}\bigcup\{K\colon K\in\mathfrak{R}\}$, and whose existence follows from Lemma 2.11. Obviously, K_0 is the desired element from $\mathfrak{R}$.

2.14. THEOREM. *Let* K_0 *be the maximal element of* $\mathfrak{R}$. *Then* $\mathrm{cl}(M\setminus K_0)$ *is an indecomposable continuum such that the union of all composants of points from* $K_0\cap\mathrm{cl}(M\setminus K_0)$ *is the whole* $\mathrm{cl}(M\setminus K_0)$.

Proof. Suppose that $\mathrm{cl}(M\setminus K_0)$ is decomposable into two proper subcontinua A and B. Then one of them, say A, must intersect K_0. The set $\mathrm{cl}(M\setminus K_0)$ is a regular-closed subset of M such that

$$\mathrm{int}[(K_0\cup A)\setminus K_0] = \mathrm{int}A\neq\emptyset.$$

But this contradicts the definition of K_0, since, in virtue of 2.11, there exists a regularly closed element in $\mathscr{K}'$ which represents the equivalence class of $K_0\cup A$, and which is not contained in K_0. Thus $\mathrm{cl}(M\setminus K_0)$ is indecomposable.

To prove the second part of the assertion take an arbitrary point p in $M\setminus K_0$. Since $p\notin E_M$, there exists a $K\in\mathscr{K}'$ such that $p\in K$. In virtue of 2.10, there is $K\,\Delta\,K_0$ and, therefore, $K_0\subset K$, K_0 being regularly closed. Now, if we take the component of p in $K\cap\mathrm{cl}(M\setminus K_0)$, then it must, obviously, intersects the set $\mathrm{Fr}K_0 = K_0\cap\mathrm{cl}(M\setminus K_0)$.

3. Main Theorem.

THEOREM. *There exists a continuum which has a connected proper and non-empty set E_M if and only if there exists an indecomposable continuum with one composant.*

Proof. Let M be a continuum which has a connected proper and non-empty set E_M. Let K_0 be a continuum defined in Theorem 2.13. Let Y be a decomposition of M consisting of the set $y_0 = K_0$ and single points of $M \setminus K_0$. Let q be the quotient map and $M' = q(M)$. To prove that M' is an indecomposable continuum suppose, on the contrary, that $M' = K \cup L$, where K and L are proper subcontinua of M'. Since q is monotone, we get

$$M' = q^{-1}(K) \cup q^{-1}(L),$$

where $q^{-1}(K)$ and $q^{-1}(L)$ are proper subcontinua. Since

$$q^{-1}(K) \cap K_0 \neq \emptyset \quad \text{and} \quad q^{-1}(L) \cap K_0 \neq \emptyset$$

(otherwise, if $q^{-1}(K) \cap K_0 = \emptyset$, then $q^{-1}(K)$ would be a proper subcontinuum of the indecomposable continuum $\mathrm{cl}(M \setminus K_0)$ and there would be $\mathrm{int}\, q^{-1}(K) \neq \emptyset$; a contradiction), we infer that

$$[K_0 \cup q^{-1}(K)] \cup [K_0 \cup q^{-1}(L)] = M$$

and

$$E_M \subset [K_0 \cup q^{-1}(K)] \cap [K_0 \cup q^{-1}(L)],$$

where $K_0 \cup q^{-1}(K)$ and $K_0 \cup q^{-1}(L)$ are proper subcontinua of M; a contradiction. We infer from 2.14 that there is only one composant in M'.

On the other hand, let K be an indecomposable continuum with one composant. Let M be the result of sticking together the arc $I = [0, 1]$ with K: by sticking, namely, the end 1 of I with an arbitrary point q of K. Let us see that E_M consists of only one point, namely, the end 0 of I. Clearly, no other point of I can be in E_M. Also points of K cannot be in E_M, for if p is a point of K, then p can be joined with q $(q = 1)$ by means of a proper subcontinuum K' of K, and we get a decomposition of M into $I \cup K'$ and K, proper subcontinua of M, having p in the intersection.

REFERENCES

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Reçu par la Rédaction le 28. 5. 1975