COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 2

RETRACTS OF THE PSEUDO-ARC

BY

J. L. CORNETTE (AMES, IOWA)

Professor B. Knaster has raised [3] the question as to whether the pseudo-arc has a non-trivial retract. Corollary 1 of this paper states that every subcontinuum of the pseudo-arc is a retract of the pseudo-arc. Descriptions and properties of the pseudo-arc may be found in [1], [2], [4], [5], and [6]. Except as noted, terminology used here related to chains is from [1] and [4] and that related to general topology is from [7].

The following theorem by R. H. Bing is of particular importance in the study of the pseudo-arc:

LEMMA 1. *Suppose* $x_1, x_2, \ldots, x_n$ *is a collection of positive integers such that* $h = x_1 \leqslant x_i \leqslant x_n = k$ *and*

$$|x_i - x_{i+1}| \leqslant 1 \quad (i = 1, 2, \ldots, n-1).$$

Suppose also that $D_1, D_2, \ldots$ *is a sequence of chains from* P *to* Q *such that for each positive integer* i, D_{i+1} *is crooked in* D_i, *the closure of each link of* D_{i+1} *is a compact subset of a link of* D_i, *and the mesh of* D_i *is less than* $1/i$. *Let* $d(i)_r$ *denote the r-th link of* D_i. *Suppose further that the subchain* $D_2(u, v)$ *of* D_2 *is contained in the subchain* $D_1(h, k)$ *of* D_1 *and the closures of* $d(2)_u$ *and* $d(2)_v$ *are mutually exclusive subsets of* $d(1)_h$ *and* $d(1)_k$ *respectively. Then for each integer* w *there is an integer* j *greater than* w *and a chain* $E = [e_1, e_2, \ldots, e_n]$ *following the pattern* $(1, x_1), (2, x_2), \ldots, (n, x_n)$ *in* D_1 *such that* E *is a consolidation of the links of* D_j *contained in* $D_2(u, v)$ *and no interior link of* E *intersects* $d(2)_u + d(2)_v$.

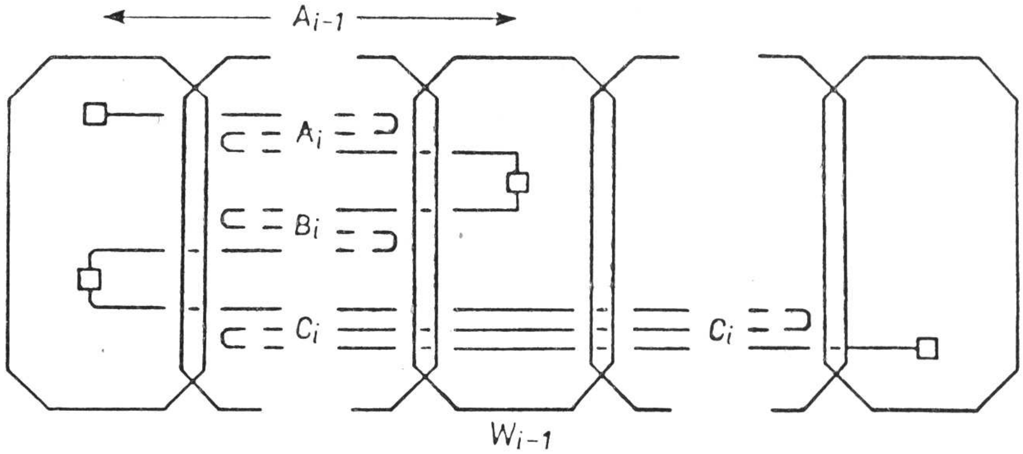
If α is a chain and β is a subchain of α, a *retraction* of α to β is a transformation γ from α to β which preserves adjacency such that if x is in β, $\gamma(x)$ is x. The *consolidation* of α induced by γ is the chain to which a link y belongs if and only if for some x in β, y is $[\gamma^{-1}(x)]^*$.

The usual description of the pseudo-arc is slightly altered. Specifically, it is assumed throughout that P and Q are two points of a compact metric space and that $W_1, W_2, \ldots$ is a sequence of chains in that space from P to Q and $A_1, A_2, \ldots$ and $B_2, B_3, \ldots$ and $C_2, C_3, \ldots$ are sequences of chains such that

(1) W_1 has nine links and A_1 is $W_1(1, 5)$.

(2) For each positive integer i, the mesh of W_i is less than $1/i$ and any link of W_i that contains a link of W_{i+1} contains the closure of that link.

(3) For each integer i greater than 1, A_i and B_i have only last links in common, B_i and C_i have only first links in common, W_i is $A_i+B_i+C_i$ (the order of B_i is reversed in W_i), A_i and B_i are crooked in A_{i-1} and have a common pattern in A_{i-1}, C_i is crooked in W_{i-1}, the first links of A_i and B_i are in only the first link of A_{i-1} (which is also the first link of W_{i-1}), and the last links of A_i and C_i are in only the last links of A_{i-1} and W_{i-1} respectively (see figure).



Let M denote the intersection of $W_1^*, W_2^*, \ldots$ and let R denote the intersection of $A_1^*, A_2^*, \ldots$ It is not difficult to verify that for each positive integer i, W_{i+1} is crooked in W_i so that M is the pseudo-arc. Furthermore, R is a non-degenerate proper subcontinuum of M and it will be shown that

THEOREM 1. *R is a retract of M.*

We must describe a continuous transformation θ from M to R such that for each x in R, $\theta(x)$ is x. In so doing, we will use methods analogous to those of Lelek in [5].

Proof of Theorem 1. We define, by induction, an infinite sequence $\{[n(i), V_i, R_i, \theta_i, E_i]\}_{i=1}^{\infty}$ such that $\{n(i)\}_{i=1}^{\infty}$ is an increasing sequence of integers with $n(1) = 1$, and for $i = 1, 2, \ldots$

(a) V_i is a consolidation of $W_{n(i)}$ which is a chain from P to Q;

(b) R_i is an initial subchain of V_i which contains $A_{n(i)}$ and (for $i > 1$) refines $A_{n(i-1)}$ — specifically, R_1 is A_1 and for $i > 1$, R_i is the chain such that Δ is a link of it if and only if for some link Δ' of $A_{n(i-1)+1}$, Δ is the sum of the links of $W_{n(i)}$ that lie in Δ';

(c) θ_i is a retraction of V_i to R_i which takes the last link of V_i to the last link of R_i.

(d) if α is a link of V_{i+1}, there is a link β of R_i such that $[\theta_i^{-1}(\beta)]^*$ contains α and β contains $\theta_{i+1}(\alpha)$; and

(e) E_i is the consolidation of V_i induced by θ_i.

For any such sequence, M is the intersection of $V_1^*, V_2^*, \ldots$ and R is the intersection of $R_1^*, R_2^*, \ldots$ In the process, additional sequences, $\{S_i\}_{i=2}^{\infty}$ and $\{T_i\}_{i=2}^{\infty}$ of chains will appear.

Initial step. Let $n(1)=1$, V_1 be W_1, R_1 be A_1 and let θ_1 be the retraction of V_1 to R_1 that takes each link of V_1-R_1 to the last link of R_1. Let E_1 be the consolidation of V_1 induced by θ_1. The parts of conditions (a)-(e) that are applicable are readily verified.

Induction step. Suppose p is a positive integer and $\{[n(i), V_i, R_i \theta_i, E_i]\}_{i=1}^{p}$ have been defined and satisfy conditions (a)-(e), where applicable. Consider Lemma 1 with the following designations:

(i) $x_1, x_2, \ldots, x_n$ describes a common pattern of $A_{n(p)+1}$ and $B_{n(p)+1}$ in R_p (such a pattern exists because R_p contains $A_{n(p)}$ and $A_{n(p)+1}$ and $B_{n(p)+1}$ have a common pattern in $A_{n(p)}$).

(ii) D_1 is E_p, D_i is $W_{n(p)+i-1}$, $i=2,3,\ldots$

(iii) $D_2(u,v)$ is $C_{n(p)+1}$ and $D_1(h,k)$ is E_p.

From the specifications in (b) and the definition of M, it follows that x_1 is 1 and x_n is the number of links in R_p. The fact that the mesh of D_1 is possibly not less than 1/1 is not significant. From (b), (c) and (e) it follows that the first link of E_p contains the first link of V_p, and from (c) and (e) it follows that the last link of E_p contains the last link of V_p. From (a) it follows that the first and last links of V_p contain, respectively, the first and last links of $W_{n(p)}$. From the definition of M, the first and last links of $W_{n(p)}$ are disjoint and contain, respectively, the closures of the first and last links of $C_{n(p)+1}$. We conclude that the closures of the first and last links of $C_{n(p)+1}$ ($d(2)_u$ and $d(2)_v$) are mutually exclusive subsets of the first and last links of E_p ($d(1)_h$ and $d(1)_k$) respectively. The remainder of the hypothesis of Lemma 1 is easily verified.

Consequently, there is an integer $n(p+1)$ greater than $n(p)$ and a chain T_{p+1} following the pattern $(1,x_1), (2,x_2), \ldots, (n,x_n)$ in E_p such that T_{p+1} is a consolidation of the links of $W_{n(p+1)}$ contained in $C_{n(p)+1}$ and no interior link of T_{p+1} intersects an end link of $C_{n(p)+1}$. Let R_{p+1} be the chain such that Δ is a link of it if and only if for some link Δ' of $A_{n(p)+1}$, Δ is the sum of the links of $W_{n(p+1)}$ that lie in Δ'. Let S_{p+1} be the chain such that Δ is a link of it if and only if Δ is the first link of T_{p+1} or for some link Δ' of $B_{n(p)+1}$ other than the first link, Δ is the sum of the links of $W_{n(p+1)}$ that lie in Δ'. Let S_{p+1} be the chain such that Δ is a link of it if and only if Δ is the first link of T_{p+1} or for some link Δ' of $B_{n(p)+1}$ other than the first link, Δ is the sum of the links of

$W_{n(p+1)}$ that lie in Δ'. Because both $A_{n(p)+1}$ and $B_{n(p)+1}$ have the pattern $(1, x_1), (2, x_2), \ldots, (n, x_n)$ in R_p, R_{p+1} has that pattern in R_p and S_{p+1} has that pattern in E_p. Let V_{p+1} be $R_{p+1}+S_{p+1}+T_{p+1}$ and let θ_{p+1} be the retraction of V_{p+1} to R_{p+1} that takes the r-th link of S_{p+1} and the r-th link of T_{p+1} to the r-th link of R_{p+1}. Let E_{p+1} be the consolidation of V_{p+1} induced by θ_{p+1}.

(a) V_{p+1} is obviously a consolidation of $W_{n(p+1)}$. The first link of V_{p+1} is the first link of R_{p+1}, contains the first link of $W_{n(p+1)}$ and therefore contains P. Q belongs to the last link of $W_{n(p+1)}$ which is contained in the last link of $C_{n(p)+1}$ and Q must then belong to a link of T_{p+1}. Because $x_1 = 1$, the first link of T_{p+1} is a subset of the first link of $C_{n(p)+1}$; from the conclusion of Lemma 1, no interior link of T_{p+1} intersects an end link of $C_{n(p)+1}$. Consequently, the last link of T_{p+1}, which is the last link of V_{p+1}, contains Q. It follows that V_{p+1} is a chain from P to Q.

(b) The specifications are contained in the definition of R_{p+1} and imply that R_{p+1} refines $A_{n(p)}$ and contains $A_{n(p+1)}$. That R_{p+1} is an initial subchain of V_{p+1} follows from the definition of V_{p+1}.

(c) By definition, θ_{p+1} is a retraction of V_{p+1} to R_{p+1} and because x_n is the number of links of R_p, θ_{p+1} takes the last link of T_{p+1} (which is the last link of V_{p+1}) to the last link of R_{p+1}.

(d) Suppose a is a link of V_{p+1}. If a is the r-th link of R_{p+1} or the r-th link of S_{p+1}, each of a and $\theta_{p+1}(a)$ (which is the r-th link of R_{p+1}) is a subset of the x_r-th link β of R_p and since $\theta_p(\beta)$ is β, $[\theta_p^{-1}(\beta)]^*$ contains a and β contains $\theta_{p+1}(a)$. If a is the r-th link of T_{p+1}, let β denote the x_r-th link of R_p. Then $[\theta_p^{-1}(\beta)]^*$ is the x_r-th link of E_p and must contain a, and β contains the r-th link of R_{p+1} which is $\theta_{p+1}(a)$.

(e) Satisfied by definition of E_{p+1}.

This completes the induction step.

Description of θ. For each point x of M and positive integer i, let $K_i(x)$ be the link or links (at most two) of v_i containing x, and let $J_i(x)$ be the sum of the elements of $\theta_i[K_i(x)]$. For x in M and $i = 2, 3, \ldots, J_i(x)$ is either a link of R_i or the sum of two intersecting links of R_i, and because R_i refines $A_{n(i-1)}$, the diameter of $J_i(x)$ is less than $2/(i-1)$. If a is in $K_{i+1}(x)$, there is a link β of R_i such that $[\theta_i^{-1}(\beta)]^*$ contains a and β contains $\theta_{i+1}(a)$, and there is a link λ of $\theta_i^{-1}(\beta)$ that contains x and hence belongs to $K_i(x)$. Then $\theta_i(\lambda)$ contains $\theta_{i+1}(a)$ and we conclude that $J_i(x)$ contains $J_{i+1}(x)$.

Consequently, for x in M, $\overline{J_1(x)}, \overline{J_2(x)}, \ldots$ is a monotonic sequence of compact sets whose diameters converge to zero and we define $\theta(x)$ to be the one point common to all of $\overline{J_i(x)}, \overline{J_2(x)}, \ldots$

θ is a retraction. If x is in R, for $i = 1, 2, \ldots, K_i(x)$ includes one or two links of R_i so that x is in $J_i(x)$. Consequently, if x is in R, $\theta(x)$ is x.

θ is continuous. Suppose ε is a positive number. Let $i > 1$ be an integer such that $3/(i-1) < \varepsilon$, and let δ be the Lebesgue number of the open cover V_i of M. Suppose x and y belong to M and the distance from x to y is less than δ. Then some link of V_i contains both x and y so that $K_i(x)+K_i(y)$ has at most three links and the diameter of $J_i(x)+J_i(y)$ is less than $3/(i-1) < \varepsilon$. Since $\theta(x)$ is in $J_i(x)$ and $\theta(y)$ is in $J_i(y)$, the distance from $\theta(x)$ to $\theta(y)$ is less than ε. It follows then that θ is continuous and the proof of Theorem 1 is complete.

COROLLARY 1. *Every subcontinuum of M is a retract of M.*

Suppose S is a subcontinuum of M. If S is degenerate or M, S is a trivial retract of M. Suppose then that S is a non-degenerate proper subcontinuum of M. From Theorem 15 of [1], there is a homeomorphism H from M to M such that $H(M)$ is M and $H(R)$ is S. Then the composition of H restricted to R and θ and H^{-1} is a retraction of M to S.

The author is indebted to Dr. A. Lelek for certain critical remarks that led to improvements in the proof of Theorem 1.

REFERENCES

[1] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Mathematical Journal 15 (1948), p. 729-742.

[2] B. Knaster, *Un continu dont tous sous-continu est indécomposable*, Fundamenta Mathematicae 3 (1922), p. 247-288.

[3] — *Problème 554*, Colloquium Mathematicum 15 (1966), p. 160.

[4] G. R. Lehner, *Extending homeomorphisms on the pseudoarc*, Transactions of the American Mathematical Society 98 (1961), p. 369-394.

[5] A. Lelek, *On weakly chainable continua*, Fundamenta Mathematicae 51 (1962), p. 271-282.

[6] E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its non-degenerate subcontinua*, Transactions of the American Mathematical Society 63 (1948), p. 581-594.

[7] R. L. Moore, *Foundations of point set theory*, Revised edition, New York 1962.

IOWA STATE UNIVERSITY, AMES, IOWA

Reçu par la Rédaction le 20. 12. 1966;
en version modifiée le 22. 8. 1967