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ON PRE-REFLEXIVE ALGEBRAS AND PRE-REFLEXIVE OPERATORS

BY

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Introduction. Let H denote an infinite-dimensional separable complex Hilbert space and let $\mathscr{B}(H)$ denote the set of all bounded operators on H. Let $\mathscr{U}$ be a weakly closed subalgebra of $\mathscr{B}(H)$ containing identity. Let

$$\mathscr{U}^* = \{A^*: A \in \mathscr{U}\}.$$

Let $\operatorname{Lat}\mathscr{U}$ denote the set of all closed linear subspaces invariant under each element of $\mathscr{U}$. We identify a closed linear subspace with the corresponding (orthogonal) projection on it. With this identification, $\operatorname{Lat}\mathscr{U}$ is the set of projections P on H such that the range of P is invariant under each element of $\mathscr{U}$. Let $(\operatorname{Lat}\mathscr{U})'$ denote the commutant of $\operatorname{Lat}\mathscr{U}$, i.e., the set of all operators commuting with each projection in $\operatorname{Lat}\mathscr{U}$. According to [1], p. 478, $\mathscr{U}$ is said to be *pre-reflexive* if

$$\mathscr{U} \cap \mathscr{U}^* = (\operatorname{Lat}\mathscr{U})'.$$

In this paper we discuss some basic algebraic properties of pre-reflexive operator algebras. Motivated by the concept of reflexive operators [3] we introduce here the notion of pre-reflexive operators and prove some of its properties. We also discuss these operators on finite-dimensional spaces.

Throughout the paper, unless otherwise stated, H stands for an infinite-dimensional separable complex Hilbert space. For a subalgebra $\mathscr{U}$ of $\mathscr{B}(H)$, by $\operatorname{Alg}\operatorname{Lat}\mathscr{U}$ we mean the following

$$\operatorname{Alg}\operatorname{Lat}\mathscr{U} = \{T: T \in \mathscr{B}(H) \text{ such that } T(M) \subseteq M \text{ for each } M \in \operatorname{Lat}\mathscr{U}\}.$$

It is clearly a weakly closed subalgebra of $\mathscr{B}(H)$. For any operator T on H, $\mathscr{A}_T$ denotes the weakly closed algebra generated by T and the identity operator I on H.

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The paper is divided into three sections. In Section 1, some properties of pre-reflexive operator algebras are obtained. Pre-reflexive operators are introduced and studied in Section 2. In Section 3, pre-reflexive operators on finite-dimensional spaces are discussed.

1. For a subalgebra $\mathcal{U}$ of $\mathcal{B}(H)$ it is easily seen that

$$\mathcal{U} \cap \mathcal{U}^* \subseteq (\operatorname{Lat} \mathcal{U})'.$$

Hence $\mathcal{U}$ is pre-reflexive if and only if

$$(\operatorname{Lat} \mathcal{U})' \subseteq \mathcal{U} \cap \mathcal{U}^*.$$

Also by [1], p. 478, $(\operatorname{Lat} \mathcal{U})'$ is the diagonal of the algebra $\operatorname{Alg}\operatorname{Lat} \mathcal{U}$, i.e.,

$$(\operatorname{Lat} \mathcal{U})' = (\operatorname{Alg}\operatorname{Lat} \mathcal{U}) \cap (\operatorname{Alg}\operatorname{Lat} \mathcal{U})^*.$$

Hence, in view of this equality, an algebra is pre-reflexive iff

$$\mathcal{U} \cap \mathcal{U}^* = (\operatorname{Alg}\operatorname{Lat} \mathcal{U}) \cap (\operatorname{Alg}\operatorname{Lat} \mathcal{U})^*. \tag{1}$$

Hence $\mathcal{U}$ is pre-reflexive if it has the same diagonal as $\operatorname{Alg}\operatorname{Lat} \mathcal{U}$. We begin our task by observing that if $\mathcal{U}$ is a reflexive operator algebra, i.e., if $\mathcal{U} = \operatorname{Alg}\operatorname{Lat} \mathcal{U}$ (see [9]), then (1) holds. Hence $\mathcal{U}$ is pre-reflexive. The converse, however, is not true (as is shown in [1], p. 504), which gives an example of a weakly closed operator algebra which contains an m.a.s.a., and hence is pre-reflexive ([10], Corollary 2) but is not reflexive. Since

$$\operatorname{Lat} \mathcal{U}^* = \{I-P\colon P \in \operatorname{Lat} \mathcal{U}\},$$

it follows that $(\operatorname{Lat} \mathcal{U})' = (\operatorname{Lat} \mathcal{U}^*)'$, so $\mathcal{U}^*$ is pre-reflexive iff $\mathcal{U}$ has this property. In addition, we have the following

1.1. THEOREM. *Any operator algebra unitarily equivalent to a pre-reflexive operator algebra is pre-reflexive.*

Proof. Let $\mathcal{U}$ be a pre-reflexive operator algebra. Let S be a unitary operator. Let

$$\mathcal{V} = S\mathcal{U}S^* = \{SAS^*\colon A \in \mathcal{U}\}$$

be the operator algebra unitarily equivalent to $\mathcal{U}$. It is sufficient to show that

$$(\operatorname{Lat} \mathcal{V})' \subseteq \mathcal{V} \cap \mathcal{V}^*.$$

Let $T \in (\operatorname{Lat} \mathcal{V})'$ and let $M \in \operatorname{Lat} \mathcal{U}$. Then $A(M) \subseteq M$ for each A in $\mathcal{U}$. Hence

$$(SAS^*)(S(M)) \subseteq S(M).$$

Therefore $S(M) \in \operatorname{Lat} \mathcal{V}$, and hence $S(M)$ reduces T since T is in $(\operatorname{Lat} \mathcal{V})'$ by assumption. This gives

$$S(M) \in \operatorname{Lat} T \cap \operatorname{Lat} T^*,$$

which further yields

$$T(S(M)) \subseteq S(M) \quad \text{and} \quad T^*(S(M)) \subseteq S(M).$$

Therefore

$$(S^* TS)(M) \subseteq M \quad \text{and} \quad (S^* T^* S)(M) \subseteq M.$$

Thus $\mathrm{Lat}\,\mathcal{U} \subseteq \mathrm{Lat}\,S^* TS$ and $\mathrm{Lat}\,\mathcal{U} \subseteq \mathrm{Lat}\,S^* T^* S$, which is equivalent to saying that $S^* TS$ and $S^* T^* S$ both belong to $\mathrm{Alg\,Lat}\,\mathcal{U}$. By taking adjoints we obtain

$$S^* TS \in (\mathrm{Alg\,Lat}\,\mathcal{U}) \cap (\mathrm{Alg\,Lat}\,\mathcal{U})^* = \mathcal{U} \cap \mathcal{U}^*$$

by hypothesis. Therefore

$$T \in S(\mathcal{U} \cap \mathcal{U}^*)S^* = (S\mathcal{U}S^*) \cap (S\mathcal{U}^*S^*)$$
$$= (S\mathcal{U}S^*) \cap (S\mathcal{U}S^*)^* = \mathcal{V} \cap \mathcal{V}^*.$$

Thus the result follows.

1.2. THEOREM. *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are pre-reflexive operator algebras, then $\mathcal{U}_1 \oplus \mathcal{U}_2$ is also pre-reflexive.*

Proof. Let

$$B \in [\mathrm{Alg\,Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2)] \cap [\mathrm{Alg\,Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2)]^*.$$

Then

$$B \in \mathrm{Alg\,Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2) \quad \text{and} \quad B^* \in \mathrm{Alg\,Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2),$$

which gives

$$\mathrm{Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2) \subseteq \mathrm{Lat}\,B \quad \text{and} \quad \mathrm{Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2) \subseteq \mathrm{Lat}\,B^*.$$

Since $\{0\} \oplus H$ and $H \oplus \{0\}$ are in $\mathrm{Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2)$, and hence in $\mathrm{Lat}\,B$, we have $B = B_1 \oplus B_2$, where

$$\mathrm{Lat}\,\mathcal{U}_1 \subseteq \mathrm{Lat}\,B_1 \quad \text{and} \quad \mathrm{Lat}\,\mathcal{U}_2 \subseteq \mathrm{Lat}\,B_2.$$

This implies that

$$B_1 \in \mathrm{Alg\,Lat}\,\mathcal{U}_1 \quad \text{and} \quad B_2 \in \mathrm{Alg\,Lat}\,\mathcal{U}_2.$$

Also then

$$\mathrm{Lat}(\mathcal{U}_1 \oplus \mathcal{U}_2) \subseteq \mathrm{Lat}\,B^* = \mathrm{Lat}(B_1 \oplus B_2)^* = \mathrm{Lat}(B_1^* \oplus B_2^*).$$

This further implies that

$$B_1 \in (\mathrm{Alg\,Lat}\,\mathcal{U}_1)^* \quad \text{and} \quad B_2 \in (\mathrm{Alg\,Lat}\,\mathcal{U}_2)^*.$$

Hence

$$B_1 \in (\mathrm{Alg\,Lat}\,\mathcal{U}_1) \cap (\mathrm{Alg\,Lat}\,\mathcal{U}_1)^* = \mathcal{U}_1 \cap \mathcal{U}_1^*,$$
$$B_2 \in (\mathrm{Alg\,Lat}\,\mathcal{U}_2) \cap (\mathrm{Alg\,Lat}\,\mathcal{U}_2)^* = \mathcal{U}_2 \cap \mathcal{U}_2^*,$$

as $\mathscr{U}_1$ and $\mathscr{U}_2$ are both pre-reflexive. By taking the direct sum we obtain

$$B = B_1 \oplus B_2 \in (\mathscr{U}_1 \cap \mathscr{U}_1^*) \oplus (\mathscr{U}_2 \cap \mathscr{U}_2^*),$$

i.e.,

$$B \in (\mathscr{U}_1 \oplus \mathscr{U}_2) \cap (\mathscr{U}_1 \oplus \mathscr{U}_2)^*.$$

Therefore $\mathscr{U}_1 \oplus \mathscr{U}_2$ is pre-reflexive.

On the same lines one can prove that the direct sum of any number of pre-reflexive algebras is pre-reflexive.

1.3. Remark. If $\mathscr{U}_1$ and $\mathscr{U}_2$ are operator algebras such that $\mathscr{U}_1 \oplus \mathscr{U}_2$ is pre-reflexive and if, in addition,

$$\operatorname{Lat}(\mathscr{U}_1 \oplus \mathscr{U}_2) = \operatorname{Lat} \mathscr{U}_1 \oplus \operatorname{Lat} \mathscr{U}_2,$$

then $\mathscr{U}_1$ and $\mathscr{U}_2$ are both pre-reflexive.

For, pre-reflexivity of $\mathscr{U}_1 \oplus \mathscr{U}_2$ implies that

$$(\mathscr{U}_1 \oplus \mathscr{U}_2) \cap (\mathscr{U}_1 \oplus \mathscr{U}_2)^* = [\operatorname{Lat}(\mathscr{U}_1 \oplus \mathscr{U}_2)]' = (\operatorname{Lat} \mathscr{U}_1)' \oplus (\operatorname{Lat} \mathscr{U}_2)'.$$

Hence

$$\mathscr{U}_1 \cap \mathscr{U}_1^* = (\operatorname{Lat} \mathscr{U}_1)' \quad \text{and} \quad \mathscr{U}_2 \cap \mathscr{U}_2^* = (\operatorname{Lat} \mathscr{U}_2)'.$$

Therefore $\mathscr{U}_1$ and $\mathscr{U}_2$ are both pre-reflexive.

We have used here the fact that if $\mathscr{U}_1$ and $\mathscr{U}_2$ are operator algebras satisfying

$$\operatorname{Lat}(\mathscr{U}_1 \oplus \mathscr{U}_2) = \operatorname{Lat} \mathscr{U}_1 \oplus \operatorname{Lat} \mathscr{U}_2,$$

then

$$\operatorname{Lat}(\mathscr{U}_1 \oplus \mathscr{U}_2)' = (\operatorname{Lat} \mathscr{U}_1)' \oplus (\operatorname{Lat} \mathscr{U}_2)'.$$

We postpone the proof of this observation to Section 3. In fact, the proof is given there for operators. For algebras, the proof follows the same lines. It can also be seen that these results can be easily extended to a finite collection of algebras.

1.4. Remark. If $\mathscr{U}$ is a pre-reflexive algebra and $\mathscr{U}^{(n)}$ is the algebra of all operators $A^{(n)} (= \sum_{i=1}^{n} \oplus A_i$, where $A_i = A$ for all i) with $A \in \mathscr{U}$, then $\mathscr{U}^{(n)}$ is pre-reflexive.

For, if

$$B \in \operatorname{Alg} \operatorname{Lat} \mathscr{U}^{(n)} \cap (\operatorname{Alg} \operatorname{Lat} \mathscr{U}^{(n)})^*,$$

i.e., if

$$\operatorname{Lat} \mathscr{U}^{(n)} \subseteq \operatorname{Lat} B \quad \text{and} \quad \operatorname{Lat} \mathscr{U}^{(n)} \subseteq \operatorname{Lat} B^*,$$

then B is of the form $\sum_{i=1}^{n} \oplus B_i$ with $\operatorname{Lat} \mathscr{U} \subseteq \operatorname{Lat} B_i$ and $\operatorname{Lat} \mathscr{U} \subseteq \operatorname{Lat} B_i^*$ for each i. This in turn implies that for each i

$$B_i \in (\operatorname{Alg} \operatorname{Lat} \mathscr{U}) \cap (\operatorname{Alg} \operatorname{Lat} \mathscr{U})^* = \mathscr{U} \cap \mathscr{U}^*,$$

as $\mathscr{U}$ is pre-reflexive. Thus we need only to show that all B_i are equal. This follows readily as the $\mathscr{U}^{(n)}$-invariant subspace $[(x, x, \ldots, x): x \in H]$ is in $\operatorname{Lat} B$, and therefore $B_i x = B_j x$ for all $x \in H$ and for every i and j.

1.5. Remark. We can also construct an operator algebra on $H \oplus H$ which is not pre-reflexive in the following way.

Let $\mathscr{U}$ denote the algebra of all operators on $H \oplus H$ of the form

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$$

with $A, B \in \mathscr{B}(H)$. Then

$$\operatorname{Alg} \operatorname{Lat} \mathscr{U} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A, B, C \in \mathscr{B}(H) \right\}.$$

Also then

$$\mathscr{U}^* = \left\{ \begin{bmatrix} A & 0 \\ B & A \end{bmatrix} : A, B \in \mathscr{B}(H) \right\},$$

which gives

$$(\operatorname{Alg} \operatorname{Lat} \mathscr{U})^* = \left\{ \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} : A, B, C \in \mathscr{B}(H) \right\}.$$

Then

$$\mathscr{U} \cap \mathscr{U}^* = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathscr{B}(H) \right\}$$

and

$$(\operatorname{Alg} \operatorname{Lat} \mathscr{U}) \cap (\operatorname{Alg} \operatorname{Lat} \mathscr{U})^* = \left\{ \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} : A, C \in \mathscr{B}(H) \right\}.$$

Hence

$$\mathscr{U} \cap \mathscr{U}^* \neq (\operatorname{Alg} \operatorname{Lat} \mathscr{U}) \cap (\operatorname{Alg} \operatorname{Lat} \mathscr{U})^*.$$

Therefore $\mathscr{U}$ is not pre-reflexive.

For the space H the tensor product of H with itself, denoted by $H \otimes H$, is the space

$$\sum_{n=1}^{\infty} \oplus H_n \quad \text{with } H_n = H \text{ for all } n.$$

This definition is a particular instance of much more general concepts of tensor product (cf. [6]). If $\mathcal{U}$ is a weakly closed subalgebra of $\mathcal{B}(H)$, then the tensor product of $\mathcal{U}$ and $\mathcal{B}(H)$, denoted by $\mathcal{U}\otimes\mathcal{B}(H)$, is the set of all operators on $H\otimes H$ of the form

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots \\ A_{21} & A_{22} & A_{23} & \dots \\ A_{31} & A_{32} & A_{33} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

such that $A_{ij}\in\mathcal{U}$ for all i and j (see [9], p. 159). In this regard we have the following

1.6. THEOREM. *If $\mathcal{U}$ is a pre-reflexive operator algebra, then $\mathcal{U}\otimes\mathcal{B}(H)$ is pre-reflexive.*

Proof. Since $\mathcal{U}$ is pre-reflexive, and hence a weakly closed algebra containing identity, $\mathcal{U}\otimes\mathcal{B}(H)$ is a weakly closed algebra of operators containing identity. Let B be an operator on $H\otimes H$ satisfying

(2) $\quad \operatorname{Lat}(\mathcal{U}\otimes\mathcal{B}(H))\subseteq \operatorname{Lat} B \quad$ and $\quad \operatorname{Lat}(\mathcal{U}\otimes\mathcal{B}(H))\subseteq \operatorname{Lat} B^*.$

Then

$$B=\begin{bmatrix} B_{11} & B_{12} & B_{13} & \dots \\ B_{21} & B_{22} & B_{23} & \dots \\ B_{31} & B_{32} & B_{33} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix},$$

where $B_{ij}\in\mathcal{B}(H)$. Suppose $M\in\operatorname{Lat}\mathcal{U}$. Let

$$N=\sum_{k=1}^{\infty}\oplus M_k \quad \text{with } M_k=M \text{ for each } k.$$

Then $N\in\operatorname{Lat}\mathcal{U}\otimes\mathcal{B}(H)$. Hence $N\in\operatorname{Lat}B$ and $N\in\operatorname{Lat}B^*$ by (2). This gives $M\in\operatorname{Lat}B_{ij}$ and $M\in\operatorname{Lat}B_{ij}^*$ for all i and j. Hence

$$\operatorname{Lat}\mathcal{U}\subseteq\operatorname{Lat}B_{ij} \quad\text{and}\quad \operatorname{Lat}\mathcal{U}\subseteq\operatorname{Lat}B_{ij}^* \quad \text{for all } i \text{ and } j.$$

Therefore, for all i and j,

$$B_{ij}\in(\operatorname{Alg}\operatorname{Lat}\mathcal{U})\cap(\operatorname{Alg}\operatorname{Lat}\mathcal{U})^*=\mathcal{U}\cap\mathcal{U}^*,$$

since $\mathcal{U}$ is pre-reflexive. This implies that B_{ij} and B_{ij}^* belong to $\mathcal{U}$ for all i and j. Hence

$$B\in\mathcal{U}\otimes\mathcal{B}(H) \quad\text{and}\quad B^*\in\mathcal{U}\otimes\mathcal{B}(H),$$

i.e.,

$$B\in(\mathcal{U}\otimes\mathcal{B}(H))\cap(\mathcal{U}\otimes\mathcal{B}(H))^*.$$

Therefore $\mathcal{U}\otimes\mathcal{B}(H)$ is pre-reflexive.

2. In this section we introduce the notion of a pre-reflexive operator in the following way:

2.1. DEFINITION. An operator T on H is said to be *pre-reflexive* if the algebra $\mathscr{A}_T$ generated by T and the identity operator is pre-reflexive. Since $\operatorname{Lat}\mathscr{A}_T = \operatorname{Lat} T$, T is pre-reflexive if

$$\mathscr{A}_T \cap \mathscr{A}_T^\ddagger = (\operatorname{Lat} T)'.$$

Every reflexive operator (i.e., an operator such that $\mathscr{A}_T$ is a reflexive algebra, see [3]) is pre-reflexive. Also a subnormal operator, an analytic Toeplitz operator and an isometry are pre-reflexive, since each of them is reflexive by [8], [11] and [4]. We also have the following fundamental results:

2.2. THEOREM. (i) *If T is pre-reflexive, then, for each scalar λ, $T+\lambda I$ is pre-reflexive.*

(ii) *If T is pre-reflexive, then T^* is also pre-reflexive.*

(iii) *Any operator unitarily equivalent to a pre-reflexive operator is pre--reflexive.*

(iv) *If T_1 and T_2 are pre-reflexive operators satisfying*

$$\mathscr{A}_{T_1\oplus T_2} = \mathscr{A}_{T_1} \oplus \mathscr{A}_{T_2},$$

then $T_1 \oplus T_2$ is pre-reflexive.

Proof. (i) Let T be pre-reflexive. Then

$$\mathscr{A}_T \cap \mathscr{A}_T^\ddagger = (\operatorname{Lat} T)'.$$

Now, since $\mathscr{A}_T = \mathscr{A}_{T+\lambda I}$ and $\operatorname{Lat} T = \operatorname{Lat}(T+\lambda I)$, we have

$$\mathscr{A}_{T+\lambda I} \cap \mathscr{A}_{T+\lambda I}^\ddagger = [\operatorname{Lat}(T+\lambda I)]'.$$

Hence $T+\lambda I$ is pre-reflexive.

(ii) The proof is the same as in the case of pre-reflexive algebras, since $(\mathscr{A}_T)^* = \mathscr{A}_{T^*}$.

(iii) Let T be pre-reflexive and let S be a unitary operator. We claim that $S\mathscr{A}_T S^* = \mathscr{A}_{STS^*}$. Since STS^* is in $S\mathscr{A}_T S^*$, we have

$$\mathscr{A}_{STS^*} \subseteq S\mathscr{A}_T S^*.$$

Also $S\mathscr{A}_T S^* = S(\mathscr{A}_{S^*STS^*S})S^*$, which is contained in $S(S^*\mathscr{A}_{STS^*}S)S^*$ by the previous argument; but that is just $\mathscr{A}_{STS^*}$, so the converse inclusion holds. Hence $S\mathscr{A}_T S^* = \mathscr{A}_{STS^*}$. Now T is pre-reflexive, therefore $\mathscr{A}_T$ is pre-reflexive, and hence, by Theorem 1.1, $S\mathscr{A}_T S^*$ is pre-reflexive. Consequently, $\mathscr{A}_{STS^*}$ is pre-reflexive, and therefore STS^* is pre-reflexive.

(iv) Let T_1 and T_2 be pre-reflexive. Then $\mathscr{A}_{T_1}$ and $\mathscr{A}_{T_2}$ are pre-reflexive. Therefore, by Theorem 1.2, $\mathscr{A}_{T_1} \oplus \mathscr{A}_{T_2}$ is pre-reflexive, i.e., $\mathscr{A}_{T_1\oplus T_2}$ is pre-reflexive. Hence $T_1 \oplus T_2$ is pre-reflexive.

2.3. Remark. If $T_1 \oplus T_2$ is pre-reflexive and if

$$\operatorname{Lat}(T_1 \oplus T_2) = \operatorname{Lat} T_1 \oplus \operatorname{Lat} T_2 \quad \text{and} \quad \mathscr{A}_{T_1 \oplus T_2} = \mathscr{A}_{T_1} \oplus \mathscr{A}_{T_2},$$

then T_1 and T_2 are both pre-reflexive.

For,

$$\mathscr{A}_{T_1 \oplus T_2} \cap (\mathscr{A}_{T_1 \oplus T_2})^* = (\operatorname{Lat}(T_1 \oplus T_2))'$$

gives

$$(\mathscr{A}_{T_1} \oplus \mathscr{A}_{T_2}) \cap (\mathscr{A}^*_{T_1} \oplus \mathscr{A}^*_{T_2}) = (\operatorname{Lat} T_1)' \oplus (\operatorname{Lat} T_2)'.$$

Therefore

$$\mathscr{A}_{T_1} \cap \mathscr{A}^*_{T_1} = (\operatorname{Lat} T_1)' \quad \text{and} \quad \mathscr{A}_{T_2} \cap \mathscr{A}^*_{T_2} = (\operatorname{Lat} T_2)'.$$

Hence T_1 and T_2 are pre-reflexive.

3. In this section we assume the space H to be finite dimensional. In this case, if T is any operator on H, the weakly closed algebra generated by T and I consists of the polynomials in T. Also then if

$$m = \prod_{i=1}^{n} p_i^{r_i}$$

is the minimum polynomial of T with $p_1, p_2, \ldots, p_n$ distinct irreducible monic polynomials, then the subspaces

$$M_i = N[P_i(T)^{r_i}], \qquad 1 \leqslant i \leqslant n,$$

where $N[P_i(T)^{r_i}]$ denotes the null space of $p_i(T)^{r_i}$, are invariant for T, are linearly independent and span H. The restrictions $T_i = T|M_i$ are called the *primary summands* of T. The representation $T = \sum_{i=1}^{n} \oplus T_i$ is called the *primary decomposition* of T (see [7], p. 180). We have the following

3.1. Theorem. *For an operator T on H, let*

$$T = \sum_{i=1}^{n} \oplus T_i \tag{3}$$

be the primary decomposition of T. Then T is pre-reflexive if and only if each T_i is pre-reflexive.

Proof. To prove the theorem we need the following lemma:

3.2. Lemma. *If T_1 and T_2 are any two operators on the Hilbert spaces H_1 and H_2 (not necessarily finite dimensional) satisfying*

$$\operatorname{Lat}(T_1 \oplus T_2) = \operatorname{Lat} T_1 \oplus \operatorname{Lat} T_2,$$

then

$$[\operatorname{Lat}(T_1 \oplus T_2)]' \doteq (\operatorname{Lat} T_1)' \oplus (\operatorname{Lat} T_2)'.$$

Supposing the validity of Lemma 3.2 we see that if T is pre-reflexive, then

$$\mathscr{A}_T \cap \mathscr{A}^*_T = (\operatorname{Lat} T)'.$$

This means, using (3), that

$$\mathscr{A}_T \cap \mathscr{A}_T^* = (\mathrm{Lat} \sum_{i=1}^{n} \oplus T_i)'.$$

Now

$$\mathrm{Lat}\, T = \mathrm{Lat} \sum_{i=1}^{n} \oplus T_i = \mathrm{Lat}\, T_1 \oplus \mathrm{Lat}\, T_2 \oplus \ldots \oplus \mathrm{Lat}\, T_n$$

by Theorem 1 of [2]. Hence, by Lemma 3.2,

$$(\mathrm{Lat}\, T)' = (\mathrm{Lat}\, T_1)' \oplus \ldots \oplus (\mathrm{Lat}\, T_n)'.$$

This gives

$$\mathscr{A}_T \cap \mathscr{A}_T^* = \sum_{i=1}^{n} \oplus (\mathrm{Lat}\, T_i)'.$$

Also

$$\mathscr{A}_T = \sum_{i=1}^{n} \oplus \mathscr{A}_{T_1}$$

by [5], p. 91. Therefore

$$(\sum_{i=1}^{n} \oplus \mathscr{A}_{T_i}) \cap (\sum_{i=1}^{n} \oplus \mathscr{A}_{T_i^*}) = \sum_{i=1}^{n} \oplus (\mathrm{Lat}\, T_1)',$$

i.e.,

$$\sum_{i=1}^{n} \oplus (\mathscr{A}_{T_i} \cap \mathscr{A}_{T_1}^*) = \sum_{i=1}^{n} \oplus (\mathrm{Lat}\, T_i)'.$$

Hence

$$\mathscr{A}_{T_i} \cap \mathscr{A}_{T_i}^* = (\mathrm{Lat}\, T_i)' \quad \text{for each } i.$$

Then each T_i is pre-reflexive. The converse can be done by retracing the steps back.

Proof of Lemma 3.2. Let

$$A \in [\mathrm{Lat}(T_1 \oplus T_2)]' = (\mathrm{Lat}\, T_1 \oplus \mathrm{Lat}\, T_2)'.$$

This means that $AP = PA$ for every projection P in $\mathrm{Lat}\, T_1 \oplus \mathrm{Lat}\, T_2$. Hence, in particular, for each projection P_1 in $\mathrm{Lat}\, T_1$ and P_2 in $\mathrm{Lat}\, T_2$,

$$A(P_1 \oplus P_2) = (P_1 \oplus P_2)A.$$

Therefore $A = A_1 \oplus A_2$, where A_1 is an operator on H_1 and A_2 is on H_2, and

$$A_1 \in (\mathrm{Lat}\, T_1)' \quad \text{and} \quad A_2 \in (\mathrm{Lat}\, T_2)'.$$

Hence

$$A = A_1 \oplus A_2 \in (\mathrm{Lat}\, T_1)' \oplus (\mathrm{Lat}\, T_2)'.$$

Thus

$$[\mathrm{Lat}(T_1 \oplus T_2)]' \subseteq (\mathrm{Lat}\, T_1)' \oplus (\mathrm{Lat}\, T_2)'.$$

The converse inclusion follows by usual computations. This proves the lemma.

If $\mathcal{U}$ is any algebra of operators on H such that each operator in $\mathcal{U}$ is pre-reflexive, then this does not necessarily imply that $\mathcal{U}$ is pre-reflexive. For example, if H is of dimension 4 and if

$$\mathcal{U} = \{T\colon\ T(c_1, c_2, c_3, c_4) = (0, 0, ac_1, bc_1 + ac_2),\ a, b \text{ scalars}\},$$

then every operator in $\mathcal{U}$ being reflexive [5] is pre-reflexive, while $\mathcal{U}$ is not pre-reflexive because the identity is not in $\mathcal{U}$. However, we have the following

3.3. THEOREM. *If $\mathcal{U}$ is a commutative pre-reflexive algebra on a finite--dimensional space, then each element of $\mathcal{U}$ is pre-reflexive.*

Proof. Let $\mathcal{U}$ be pre-reflexive and $T \in \mathcal{U}$. Let

$$B \in (\mathrm{Alg\,Lat}\, T) \cap (\mathrm{Alg\,Lat}\, T)^*,$$

i.e.,

$$\mathrm{Lat}\, T \subseteq \mathrm{Lat}\, B \quad \text{and} \quad \mathrm{Lat}\, T \subseteq \mathrm{Lat}\, B^*.$$

Since $T \in \mathcal{U}$, we have $\mathrm{Lat}\,\mathcal{U} \subseteq \mathrm{Lat}\, T$. Consequently,

$$\mathrm{Lat}\,\mathcal{U} \subseteq \mathrm{Lat}\, B \quad \text{and} \quad \mathrm{Lat}\,\mathcal{U} \subseteq \mathrm{Lat}\, B^*.$$

This implies that

$$B \in \mathrm{Alg\,Lat}\,\mathcal{U} \cap (\mathrm{Alg\,Lat}\,\mathcal{U})^*,$$

and therefore $B \in \mathcal{U} \cap \mathcal{U}^*$, as $\mathcal{U}$ is pre-reflexive. Now $B \in \mathcal{U}$, $T \in \mathcal{U}$ and $\mathcal{U}$ is commutative, so $BT = TB$. Thus T and B are operators on a finite-dimensional space such that $B \in \mathrm{Alg\,Lat}\, T$ and B commutes with T; therefore B is a polynomial in T ([2], Theorem 10). This gives $B \in \mathcal{A}_T$. Also $B^* \in \mathcal{U}$ and $B^* \in \mathrm{Alg\,Lat}\, T$ imply that B^* is a polynomial in T; therefore $B^* \in \mathcal{A}_T$ by the same argument. Thus $B \in \mathcal{A}_T \cap \mathcal{A}_T^*$. Hence T is pre-reflexive.

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