COLLOQUIUM MATHEMATICUM

VOL. XXV 1972 FASC. 2

ON SOME INTEGRAL EQUATION

BY

J. ZABCZYK (WARSZAWA)

Let us consider a flow $x(s)$, $-\infty < s < +\infty$, which is supplied by an input flow $y(s)$ and yields an output flow $\big(1-a(s)\big)x(s)$, where $a(s)$ is a given function with $0 \leqslant a(s) \leqslant 1$. The remainder $a(s)x(s)$ of the flow $x(s)$ goes into a container from which it returns to the flow $x(s)$ in a way such that the amount k which remains in the container at the moment t is ruled by a function φ in a way that

$$k = \int_{t-u}^{t} a(s)x(s)\varphi(t-s)\,ds,$$

where u is a positive constant.

Following this description, the function φ should be assumed to be non-increasing and to satisfy the condition $0 \leqslant \varphi(s) \leqslant 1$, $s \in (-\infty, +\infty)$. Since there are no other sinks or sources in the system besides mentioned above, we should have at any moment t the equality

$$(1) \qquad \int_{-\infty}^{t} \big(1-a(s)\big)x(s)\,ds + \int_{t-u}^{t} a(s)x(s)\varphi(t-s)\,ds = \int_{-\infty}^{t} y(s)\,ds.$$

This raises the more general problem: under what conditions on y, a and φ, is x determined uniquely by equation (1)?

Primarily this problem was considered by J. Łoś in connection with "flows" of goods in a closed economy. In that case y is the flow of labour, and x is the flow of national income divided by steering function a into consumption $\big(1-a(s)\big)x(s)$ and investments $a(s)x(s)$. Then K is capital working at the moment t. All flows are expressed in the same units of "value", e.g., in workers per squared hour. Thus integrated quantities are expressed in workers per hour.

The theorem, which we are going to prove as a partial solution to the problem, is the following

THEOREM. *Let a be a measurable function defined on $(-\infty, +\infty)$ and such that*

$$0 \leqslant a_0 \leqslant a(s) \leqslant a_1 \leqslant 1 \quad \textit{for every } s \leqslant t_0.$$

Let φ be a function on $[0, u]$, non-negative, non-increasing and of the class C^1. If

$$\varphi(0) \leqslant 1 \quad \text{and} \quad a_1 < 1 \tag{2}$$

or

$$\varphi(0) > 1 \quad \text{and} \quad \varphi(0)(a_1 - a_0) < 1 - a_0, \tag{2'}$$

then for every $y \in L(-\infty, t_0)$ there exists a unique function $x \in L(-\infty, t_0)$ such that equality (1) holds true for every $t \leqslant t_0$. Moreover, if y is non-negative, then also x is non-negative.

Proof. Differentiating (1) we obtain for almost all $t < t_0$ an equivalent equation

$$\big(1 - a(t)\big)x(t) + \int_{t-u}^{t} a(s)x(s)\varphi'(t-s)\,ds + \tag{3}$$
$$+ a(t)x(t)\varphi(0) - a(t-u)x(t-u)\varphi(u) = y(t),$$

which may be re-written in the form

$$x - A_1 x - A_2 x = \bar{y},$$

where, for $t \in (-\infty, t_0)$,

$$x = x(t), \quad \bar{y} = \frac{1}{1 - a(t) + \varphi(0)a(t)},$$

and

$$A_1 x = \frac{-1}{1 - a(t) + \varphi(0)a(t)} \int_{t-u}^{t} a(s)x(s)\varphi'(t-s)\,ds,$$

$$A_2 x = \frac{\varphi(u)a(t-u)}{1 - a(t) + \varphi(0)a(t)} x(t-u).$$

Following our assumptions we have

$$1 - a(t) + \varphi(0)a(t) \geqslant \begin{cases} (1 - a_1) + a_1\varphi(0) > 0 & \text{if } \varphi(0) \leqslant 1, \\ 1 + a_0\big(\varphi(0) - 1\big) > 0 & \text{if } \varphi(0) > 1. \end{cases}$$

Both operators A_1 and A_2 transform the space $L(-\infty, t_0)$ into itself and transform a non-negative function into a non-negative one. Moreover, the two operators are bounded, since

$$\|A_1 x\| = \int_{-\infty}^{t_0} \left| \frac{1}{1 - a(t) + \varphi(0)a(t)} \int_{t-u}^{t} a(s)x(s)\varphi'(t-s)\,ds \right| dt \tag{4}$$
$$\leqslant \sup_{t \leqslant t_0} \frac{1}{1 - a(t) + \varphi(0)a(t)} \int_0^u \int_{-\infty}^{t_0} a(t-s)\,|x(t-s)|\,|\varphi'(s)|\,ds\,dt$$
$$\leqslant \sup_{t \leqslant t_0} \frac{a_1}{1 - a(t) + \varphi(0)a(t)} \int_0^u |\varphi'(s)|\,ds\,\|x\|$$
$$\leqslant \sup_{t \leqslant t_0} \frac{a_1\big(\varphi(0) - \varphi(u)\big)}{1 - a(t) + \varphi(0)a(t)} \|x\|$$

and

$$\|A_2x\| \leqslant \sup_{t\leqslant t_0} \frac{\varphi(u)\,a_1}{1-a(t)+\varphi(0)\,a(t)}\,\|x\|. \tag{5}$$

It follows from (2) that

$$\|A_1+A_2\| \leqslant \|A_1\|+\|A_2\| \leqslant \frac{a_1\big(\varphi(0)-\varphi(u)\big)}{1-a_1+\varphi(0)\,a_1}+\frac{a_1\varphi(u)}{1-a_1+\varphi(0)\,a_1},$$

$$\frac{a_1\varphi(0)}{1-a_1+a_1\varphi(0)}<1$$

and from (2′) that

$$\|A_1+A_2\| \leqslant \frac{\varphi(0)\,a_1}{1-a_0+\varphi(0)\,a_0}<1.$$

The application of the fixed-point theorem for contracting operators(¹) completes the proof.

Remark 1. It is a routine matter to generalize the theorem to functions φ which are only piecewise of the class C^1 on $[0, u]$, where $0 < u \leqslant +\infty$.

Remark 2. Our theorem covers the case of $a(s) = a_0 = a_1 < 1$. Equation (1) may then be put in the form

$$\int_{-\infty}^{t} x(s)\,ds+\int_{t-u}^{t} x(s)\bar{\varphi}(t-s)\,ds = \int_{-\infty}^{t} y(s)\,ds \tag{6}$$

with $\bar{\varphi} = \varphi/(1-a_1)$.

Remark 3. Conditions (2) or (2′) are essential for the validity of the conclusion. If

$$\big(1-a(s)\big) \leqslant \frac{C}{|s|} \qquad \text{for } s\leqslant t_0 \text{ and a constant } C,$$

then the left-hand side of (1) is in $L(-\infty, t_0)$, but the right-hand side need not be in that space.

Remark 4. If the function φ is either not non-increasing or non-negative, then equation (1) may have many solutions.

To construct an example, let us consider the particular form (6) of equation (1) and let us assume $y(s) = 0$ and

$$\bar{\varphi}(t) = \begin{cases} 0 & \text{if } 0\leqslant t\leqslant \frac{u}{2}, \\ c & \text{if } \frac{u}{2}<t<u. \end{cases}$$

(¹) И. Г. Петровский, *Лекции по теории обыкновенных дифференциальных уравнений*, Москва 1964, p. 58-64.

Then (6) is equivalent to the following equation:

$$x(t)+cx\left(t-\frac{u}{2}\right)-cx(t-u) = 0. \tag{7}$$

If $c = (e^{-u/2}+e^{-u})^{-1}$, then every function $x(t) = e^t f(t)$, where f is a periodic function with $f(t) = f(t-u) = -f(t-u/2)$, satisfies (7).

If $c = (e^{-u}-e^{-u/2})^{-1}$, then every function $x(t) = e^t f(t)$, where $f(t) = f(t-u/2)$, satisfies (7).

The author wishes to express his thanks to Professor J. Łoś for suggesting the problem and much help while working on it.

Reçu par la Rédaction le 29. 9. 1970