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ON THE PROXIMATE FIXED-POINT PROPERTY FOR MULTIFUNCTIONS

BY

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In this paper * a study of ε-continuity is initiated from the viewpoint of multifunctions, and a proximate fixed-point property is developed for various classes of ε-continuous multifunctions. The purpose of this paper is to generalize some fixed-point theorems of Klee [2] and Yandl [7] to the setting of multifunctions. The main results are that if a metric space X has the proximate fixed-point property for multifunctions, then so does every compact m-retract of X and that if, further, X is compact, then every metric homeomorph of X has the proximate fixed-point property for multifunctions. The author and R. E. Smithson have proved in related papers [4], [5] that non-empty, compact, convex subspaces of locally convex, Hausdorff linear topological spaces and trees both have the proximate fixed-point property for various classes of multifunctions.

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A *multifunction* F on a space X into a space Y is a correspondence between elements of X and non-empty subsets of Y. To be precise $F \subset X \times Y$, and, for each $x \in X$, $\pi_2\big((\{x\} \times Y) \cap F\big) \neq \square$ where π_2 is the second projection on $X \times Y$ into Y and $\square$ denotes the empty set. In particular, every (single-valued) function is a multifunction. We shall write $F\colon X \to Y$ for a multifunction on X into Y and $F(x)$ for the set $\pi_2\big((\{x\} \times Y) \cap F\big)$. If $A \subset X$, then $F(A) = \bigcup\{F(x) \mid x \in A\}$. If U is a subset of some topological space X, then U° and U^* denote the interior and closure, respectively, of U in X.

DEFINITION. Let $F\colon X \to Y$. Then F is *lower semi-continuous* (l.s.c.) if and only if for each $x \in X$ and for each $V = V^\circ \subset Y$ such that $F(x) \cap V \neq \square$ there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(x') \cap V \neq \square$ for all $x' \in U$. Further, F is *upper semi-continuous* (u.s.c.) if and only if for each $x \in X$ and for each $V = V^\circ \subset Y$ such that $F(x) \subset V$ there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(U) \subset V$. The multifunction F is *continuous* if and only if F is both l.s.c. and u.s.c.

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If r is a positive real number and if A is a subset of a metric space X, then

$$S_r(A) = \{x \mid x \in X \text{ and } d(x, A) < r\}.$$

DEFINITION. Let $F: X \to Y$, where Y is a metric space, and let $\varepsilon > 0$. Then F is *lower ε-continuous* (l.ε-c.) if and only if for each $x \in X$ and for each $y \in F(x)$ there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(x') \cap S_\varepsilon(y) \neq \square$ for all $x' \in U$. Further, F is *upper ε-continuous* (u.ε-c.) if and only if for each $x \in X$ there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(U) \subset S_\varepsilon(F(x))$. The multifunction F is *ε-continuous* if and only if F is both l.ε-c. and u.ε-c.

Let P be a property of sets. Then $F: X \to Y$ is *point P* (*image P*) if and only if $F(x)$ has property P for each $x \in X$ ($F(E)$ has property P for each $E \subset X$ with property P).

An alternate definition of ε-continuity for point closed multifunctions $F: X \to Y$, where Y is a metric space of finite diameter, can be formulated in terms of the Hausdorff metric [1] for the space $S(Y)$ of non-empty, closed subsets of Y and the induced function $f: X \to S(Y)$. If, moreover, F is point compact, then the two definitions of ε-continuity are essentially equivalent [3].

We now indicate the close correspondence between continuity and ε-continuity. It follows from the above definitions that if $F: X \to Y$ is point compact and u.ε-c. for all $\varepsilon > 0$, then F is u.s.c. Furthermore, if $F: X \to Y$ is point compact, then F is ε-continuous for all $\varepsilon > 0$ if and only if F is continuous.

DEFINITION. Let $A \subset X$. Then A is an *m-retract* of X if and only if there exists a continuous multifunction $F: X \to A$ such that $F(x) = \{x\}$ for all $x \in A$.

The proof of the following lemma is not difficult:

LEMMA 1. *Let A be a compact subset of a metric space X, let $F: X \to A$ be an u.s.c. multifunction such that $F(x) = \{x\}$ for all $x \in A$, and let $\eta > 0$. Then there is a $\lambda > 0$ such that if $x \in X$ and if $d(x, A) < \lambda$, then $d(x, F(x)) < \eta$ and $d(F(x)) < \eta$.*

If $F: X \to Y$ and if $G: Y \to Z$, then a multifunction $G \circ F: X \to Z$ is defined by $G \circ F(x) = G(F(x))$ for each $x \in X$.

LEMMA 2. *If $F: X \to Y$ is ε-continuous where Y is a subspace of a metric space Z and if $G: Y \to Z$ is such that $d(y, z) \leqslant \varepsilon_0$ for all $y \in Y$ and for all $z \in G(y)$, then $G \circ F: X \to Z$ is ε'-continuous where $\varepsilon' = \varepsilon + 2\varepsilon_0$.*

Proof. In order to show lower ε'-continuity, let $x \in X$ and let $z \in G \circ F(x)$. Then there exists $y \in F(x)$ such that $z \in G(y)$. Since F is l. ε-c. there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(x') \cap S_\varepsilon(y) \neq \square$ for all $x' \in U$. Let $x' \in U$ and choose $y' \in F(x')$ such that $d(y, y') < \varepsilon$. Then we have $d(z, G(y')) \leqslant d(z, y)+d(y, y')+d(y', G(y')) < \varepsilon+2\varepsilon_0 = \varepsilon'$. Therefore $G \circ F(x') \cap S_{\varepsilon'}(z) \neq \square$, whence $G \circ F$ is l.ε'-c. The proof of upper ε'-continuity is entirely similar.

LEMMA 3. *If $F: X \to Y$ is continuous and if $G: Y \to Z$ is ε-continuous, then $G \circ F: X \to Z$ is ε-continuous.*

Proof. The proof of lower ε-continuity is straightforward. For upper ε-continuity let $x \in X$. Then for each $y \in F(x)$ there exists $V_y = V_y^\circ \subset Y$ with $y \in V_y$ such that $G(V_y) \subset S_\varepsilon(G(y))$ since G is u.ε-c. As F is u.s.c. there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(U) \subset \cup \{V_y | y \in F(x)\}$. It is clear that $G \circ F(U) \subset S_\varepsilon(G \circ F(x))$, whence $G \circ F$ is u.ε-c.

DEFINITION. A metric space X has the *proximate fixed-point property for multifunctions* (p.F.p.p.) if and only if for each $\eta > 0$ there is an $\varepsilon > 0$ such that for every ε-continuous multifunction $F: X \to X$ there is a point $x \in X$ such that $d(x, F(x)) < \eta$.

The proximate fixed-point properties for single-valued functions (p.f.p.p.), for u.ε-c. multifunctions, and for multifunctions with restrictions on the image sets are defined analogously.

THEOREM 1. *If a metric space X has the p.F.p.p., then every compact m-retract of X has the p.F.p.p.*

Proof. Let A be a compact m-retract of X and let $\eta > 0$. Then there exists a continuous multifunction $F: X \to A$ such that $F(x) = \{x\}$ for all $x \in A$. By Lemma 1 there is a $\lambda \in (0, \eta/8)$ such that if $x \in X$ and if $d(x, A) < \lambda$, then $d(x, F(x)) < \eta/4$ and $d(F(x)) < \eta/4$. Since X has the p.F.p.p. there is an $\varepsilon > 0$ such that for every ε-continuous multifunction $H: X \to X$ there is a point $x \in X$ such that $d(x, H(x)) < \lambda$. The claim is that ε works for A also. For let $G: A \to A$ be ε-continuous and let i be the inclusion on A into X. Lemmas 2 and 3 readily imply that $i \circ (G \circ F)$: $X \to X$ is ε-continuous. Thus there is a point $x \in X$ such that $d(x, G \circ F(x)) < \lambda$. Consequently $d(x, A) < \lambda$, and therefore $d(x, F(x)) < \eta/4$ and $d(F(x)) < \eta/4$. Now we have $d(F(x), G \circ F(x)) \leqslant d(F(x), x)+d(x, G \circ F(x)) < \eta/4+\lambda < \eta/2$. Accordingly there exist points $(p, r) \in F(x) \times F(x)$ and $q \in G(p)$ such that $d(r, q) < \eta/2$. Thus $d(p, G(p)) \leqslant d(p, q) \leqslant d(p, r)+d(r, q) < d(F(x))+\eta/2 < \eta$, and A has the p.F.p.p.

The above proof of Theorem 1 follows to a fashion of Klee's proof of the assertion of Theorem 1 for single-valued functions [2].

DEFINITION. A space X has the *fixed-point property for multifunctions* (F.p.p.) if and only if for each continuous multifunction $F: X \to X$ there is a point $x \in X$ such that $x \in F(x)$.

The fixed-point properties for single-valued functions (f.p.p.), for u.s.c. multifunctions, and for multifunctions with restrictions on the image sets are defined similarly.

LEMMA 4. *If a compact metric space X has the p.F.p.p., then X has the F.p.p. for point-closed multifunctions.*

Proof. Let $F: X \to X$ be continuous and point closed. It is immediate that $G(F) = \bigcup\{(x, y) \mid x \in X \text{ and } y \in F(x)\}$ is a compact subset of $X \times X$. Therefore the function $d|_{G(F)}: G(F) \to R$, where R denotes the real numbers, assumes a minimum value r on $G(F)$. If $d(x, F(x)) > 0$ for each $x \in X$, then $r > 0$. Let $\eta = r/2$. Since F is ε-continuous for every $\varepsilon > 0$, there does not exist an $\varepsilon > 0$ such that for every ε-continuous multifunction $G: X \to X$ there is a point $x \in X$ such that $d(x, G(x)) < \eta$.

A proof of Lemma 4 employing nets appears in [4] and [5]. Yandl [7] proved Lemma 4 for single-valued functions. Partial converses to this lemma may be found in [3] and [4].

COROLLARY. *If a metric space X has the p.F.p.p., then every compact m-retract of X has the F.p.p. for point closed multifunctions.*

Let E^2 be a Euclidean 2-cell. By a theorem of Klee [2], every compact, metric absolute retract has the p.f.p.p. As a consequence E^2 has the p.f.p.p. but lacks the p.F.p.p. since Strother [6] has exhibited a continuous, point closed multifunction on E^2 into itself which does not have a fixed point.

LEMMA 5. *Let $g: Y \to Z$ be a continuous (single-valued) function on a compact metric space Y into a metric space Z and let $\varepsilon' > 0$. Then there is an $\eta > 0$ such that if $F: X \to Y$ is ε-continuous where $0 < \varepsilon \leqslant \eta$, then $g \circ F: X \to Z$ is ε'-continuous.*

Proof. Since Y is compact and g is continuous, there is an $\eta > 0$ such that if $(y, y') \in Y \times Y$ and if $d(y, y') < \eta$, then $d(g(y), g(y')) < \varepsilon'$. Let $\varepsilon \in (0, \eta]$ and let $F: X \to Y$ be ε-continuous. To prove the lower ε'-continuity of $g \circ F$, let $x \in X$ and let $z \in g \circ F(x)$. Then there is a point $y \in F(x)$ such that $g(y) = z$. Since F is l.ε-c. there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(x') \cap S_\varepsilon(y) \neq \square$ for all $x' \in U$. Hence $g \circ F(x') \cap S_{\varepsilon'}(z) \neq \square$ for all $x' \in U$. Therefore $g \circ F$ is l.ε'-c. For upper ε'-continuity let $x \in X$. Since F is u.ε-c. there exists $U = U^\circ \subset X$ with $x \in U$ such that $F(U) \subset S_\varepsilon(F(x))$. It follows that $g \circ F(U) \subset S_{\varepsilon'}(g \circ F(x))$. Therefore $g \circ F$ is u.ε'-c., and hence ε'-continuous.

THEOREM 2. *If a compact metric space X has the p.F.p.p., then every metric homeomorph of X has the p.F.p.p.*

Proof. Let $h: X \to Y$ be a homeomorphism of X onto a metric space Y and let $\eta > 0$. Since X is compact and h is continuous, there is an $\eta' > 0$ such that if $(x, x') \in X \times X$ and if $d(x, x') < \eta'$, then $d(h(x), h(x')) < \eta$. Since X has the p.F.p.p. there is an $\varepsilon' > 0$ such that for every ε'-continuous multifunction $F: X \to X$ there is a point $x \in X$ such that $d(x, F(x)) < \eta'$. Since Y is compact and $h^{-1}: Y \to X$ is a continuous function, there exists an $\varepsilon > 0$ by Lemma 5 such that if $F': X \to Y$ is

ε-continuous, then $h^{-1} \circ F' \colon X \to X$ is ε'-continuous. Let $G \colon Y \to Y$ be an arbitrary ε-continuous multifunction. Lemmas 3 and 5 imply that $h^{-1} \circ (G \circ h) \colon X \to X$ is ε'-continuous. Thus there is a point $x \in X$ such that $d\big(x, h^{-1} \circ (G \circ h)(x)\big) < \eta'$, whence $d\Big(h(x), G\big(h(x)\big)\Big) < \eta$ and Y has the p.F.p.p.

It can be seen by means of an inductive proof that $(0, 1)$ and $[0, 1]$ both have the p.F.p.p. for point compact multifunctions. Therefore the p.F.p.p. is not a topological invariant since the set of real numbers clearly does not have the p.F.p.p.

We now prove a generalization of the last theorem in Klee's paper [2].

THEOREM 3. *Let X be a compact Hausdorff space which is an absolute retract for such spaces. Then for each open cover $\mathscr{U}$ of X there exists a finite open cover $\mathscr{V}$ of X which has the following property: if $G \colon X \to X$ is any multifunction such that for each $x \in X$ there exists $N_x = N_x^\circ \subset X$ with $x \in N_x$ satisfying $G(N_x) \subset V$ for some $V \in \mathscr{V}$, then there is a point $x_0 \in X$ such that $x_0 \in U$ and $G(x_0) \subset U$ for some $U \in \mathscr{U}$.*

Proof. We can assume that X is a compact retract of a Tychonoff cube $T = [0, 1]^M$, and we consider T as a subset of the linear topological space R^M, where R denotes the real numbers. Let $\mathscr{B}$ be a symmetric base for the uniformity for T such that for each $x \in T$ and for each $B \in \mathscr{B}$, we have that $B[x]$ is convex. Suppose that $\mathscr{U}$ is an open cover of X. Then there exists a member R of the uniformity for T such that $R[x]$ is a subset of some member of $\mathscr{U}$ for every $x \in X$. Choose $\hat{B} \in \mathscr{B}$ such that $\hat{B} \circ \hat{B} \subset R$, and pick a continuous function $f \colon T \to X$ satisfying $f(x) = x$ for all $x \in X$. There exists $\bar{B} \in \mathscr{B}$ with $\bar{B} \subset \hat{B}$ such that if $t \in T$ and if $\bar{B}[t] \cap X \neq \Box$, then $f(t) \in \hat{B}[t]$. This follows from an extension of Lemma 1. Choose $B \in \mathscr{B}$ such that $B \circ B \subset \bar{B}$. Then we let $\mathscr{V}$ be a finite subcover of $\{B[x] \mid x \in X\}$ which covers X. Suppose that $G \colon X \to X$ is any multifunction such that for each $x \in X$ there exists $N_x = N_x^\circ \subset X$ with $x \in N_x$ satisfying $G(N_x) \subset V$ for some $V \in \mathscr{V}$. Now choose a finite subcover $\mathscr{N}$ of $\{N_x \mid x \in X\}$ which covers X, and for each $x \in X$ let $W_x = \cap \{N_{x'} \mid x \in N_{x'} \in \mathscr{N}\}$. For each $S \subset X$ let $CH(S)$ denote the convex hull of S in T. Then define a multifunction $H \colon X \to T$ by $H(x) = \Big(CH\big(G(W_x)\big)\Big)^*$ for each $x \in X$. The multifunction H is obviously point closed, point convex, and u.s.c. By the Kakutani-Fan-Glicksberg fixed-point theorem [1] (which is generalized in [3] and [4]), there is a point $t_0 \in T$ such that $t_0 \in H \circ f(t_0)$. Let $x_0 = f(t_0)$. By the definition of H there is a point $t_1 \in CH\big(G(W_{x_0})\big)$ such that $t_1 \in B[t_0]$. Since $\mathscr{N}$ covers X and since $B[x]$ is convex for each $x \in T$, there is a point $(x, x') \in X \times X$ such that $CH\big(G(W_{x_0})\big) \subset CH\big(G(N_{x'})\big) \subset B[x]$. Thus $(t_0, x) \in \bar{B}$, whence $(t_0, x_0) \in \hat{B}$. Therefore $(x, x_0) \in R$. But also $G(x_0) \subset B[x] \subset R[x]$, and this proves the theorem.

Remarks. Theorems 1 and 2 are easily generalized to uniform spaces. Furthermore Theorem 1 can be extended to a generalization of a theorem of Yandl [7] on strong proximate retracts. It is only necessary to use Lemma 1 as a guide to defining the concept of a strong proximate m-retract thereby obtaining directly an extension of Theorem 1 and Yandl's theorem. The details are in [3]. Variants of Theorems 1 and 2 are of course immediate by restricting the classes of multifunctions considered. We indicate in closing an outgrowth of Theorem 3. Suppose that X is a compact Hausdorff space which has a symmetric base $\mathscr{U}_0$ for its uniformity. For a given property P of sets, let $P(X)$ be the subsets of X which have property P. Suppose further that there is a function $K\colon 2^X \to 2^X$ (where 2^X denotes the collection of all subsets of the set X) such that P and K satisfy the following conditions:

(i) If $A \epsilon 2^X$, then $A \subset K(A) \epsilon P(X)$.

(ii) If $A \subset B \epsilon 2^X$, then $K(A) \subset K(B)$.

(iii) If $A \epsilon P(X)$, then $K(A) = A$ and $A^* \epsilon P(X)$.

(iv) $\{\{x\} \mid x \epsilon X\} \cup \{\square\} \subset P(X)$.

(v) If $A \epsilon P(X)$ and if $U \epsilon \mathscr{U}_0$, then $U[A] \epsilon P(X)$.

We call K a P-operator for X. If, in addition to the above assump, tions, X has the F.p.p. for point closed, point P, u.s.c. multifunctions- then the conclusion of Theorem 3 is verified. The proof of this assertion duplicates the proof of Theorem 3 given above. Two examples of spaces X with the aforementioned properties are non-empty, compact, convex subsets of locally convex, Hausdorff linear topological spaces and hereditarily unicoherent, arcwise connected, locally connected continua (trees) [4].

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