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CONCERNING THREE QUESTIONS OF BURGESS ABOUT HOMOGENEOUS CONTINUA

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Burgess [6] has raised the following three questions:

If every proper subcontinuum of a metric continuum M is homogeneous, is M itself homogeneous?

For each positive integer n, does there exist a homogeneous plane continuum which separates the plane into n connected domains?

Does there exist a homogeneous plane continuum having infinitely many complementary domains?

In this note it is shown that, for each positive n and for $n = \infty$ as well, there exists a plane continuum with n complementary domains each non-degenerate subcontinuum of which is a pseudo-arc. Hence (cf. [2]), if the answer to the first question of Burgess is in the affirmative, then so is the answer to the other two.

Definitions. If C and D are circular chains, then D is said to be *crooked* in C provided that the closure of each link of D is a subset of some link of C and, if C' and D' are simple chains which are subcollections of C and D respectively, and the closure of each link of D' is a subset of some link of C', then D' is crooked in C' (cf. [2] or [7]).

If, in the plane S, n is a positive integer and K is a set of only n points, then the coherent collection H of domains will be said to be K-*admissible* provided that

(1) $\bar{H}^*$ is, topologically, a disk with $n-1$ holes and separates each two points of K from each other in S;

(2) H contains a domain J, called the *joiner* of H; and

(3) H is the sum of $n-1$ circular chains $D_1, \ldots, D_{n-1}$ such that if i and j are two integers $(1 \leqslant i < j \leqslant n-1)$, then $D_i^* \cap D_j^* = J$ and $D_i \cap D_j = \{J\}$.

THEOREM 1. *If n is a positive integer, there exists, in the plane S, a continuum M with n complementary domains such that each non-degenerate proper subcontinuum of M is a pseudo-arc.*

Proof. Let K denote a set of only n points. Let $H_0, H_1, \ldots$ denote a sequence of K-admissible finite coherent collections of domains such that, for each i,

(1) mesh $(H_i) < 1/(i+1)$;

(2) the closure of each element of H_{i+1} is a subset of some element of H_i;

(3) if L is an element of H_i, there exists an integer $j > i$ such that the joiner of H_j is a subset of L; and

(4) if i is even, the joiner of H_{i+1} is a subset of the joiner of H_i and, if C is a circular chain which is a subcollection of H_i, then there is a circular chain D in H_{i+1} which is crooked in C.

Let $M = \bigcap_{i>0} H_i^*$. Clearly, M has only n complementary domains, each containing a point of K.

Suppose that N is a non-degenerate proper subcontinuum of M and ε is a positive number. Let i be an integer greater than $1/\varepsilon$ such that some element L of H_i does not intersect N. Then there exists an integer $j > i$ such that the joiner J of H_j is a subset of L. Then $H_j - \{J\}$ is the sum of ε-chains one of which, C, covers N; hence, N is chainable. There exists a positive integer $k > j$ such that the joiner of H_k is a subset of J and the collection D of all elements of H_k which intersect N is a simple chain which is crooked in C. Hence, N is a pseudo-arc, [3].

THEOREM 2. *There exists, in the plane S, a continuum M with infinitely many complementary domains such that each non-degenerate proper subcontinuum of M is a pseudo-arc.*

Proof. Let $K_0, K_1, \ldots$ denote a sequence of finite point sets and $H_0, H_1, \ldots$ denote a sequence of finite coherent collections of domains such that K_0 contains three points and, for each n,

(1) K_n is a subset of K_{n+1};

(2) each point of $K_{n+1} - K_n$ is in H^*_{2n+1};

(3) mesh $(H_i) < 1/(i+1)$;

(4) H_{2n} and H_{2n+1} are K_n-admissible;

(5) the closure of each element of H_{i+1} is a subset of some element of H_i;

(6) if L is an element of H_i, there exists an integer $j > i$ such that the joiner of H_j is a subset of L; and

(7) if i is even, the joiner of H_{i+1} is a subset of the joiner of H_i and, if C is a circular chain which is a subcollection of H_i, then there is a circular chain D in H_{i+1} which is crooked in C.

Let $M = \bigcap_{i>0} H_i^*$. Clearly, M has infinitely many complementary domains. That each non-degenerate proper subcontinuum of M is a pseudo-arc follows from an argument similar to that for Theorem 1.

Remark. Ingram [9] has shown that if each proper subcontinuum of the decomposable metric continuum K is chainable, then K is either chainable or circle-like. The continuum M of Theorem 2 is neither chainable nor circle-like, since $S-M$ has more than two components, [4].

Burgess [5] has shown that, if the bounded plane continuum M is nearly homogeneous and has a finite number of complementary domains, then M is the boundary of each of its complementary domains. Each of the continua of Theorem 1 and Theorem 2 is the boundary of each of its complementary domains.

THEOREM 3. *Suppose that M is a plane continuum no one of whose proper subcontinua separates the plane. Then M is the boundary of each of its complementary domains.*

Proof. If the boundary β of the complementary domain D of M is a proper subcontinuum of M, then β separates the plane.

The study of homogeneous continua seems in some way tied to the question of which continua are homeomorphic to each of their non-degenerate subcontinua. In the plane, each such continuum is tree-like, [10]. If such a continuum is chainable, it is either an arc, [8], or a pseudo-arc, [3]. That it may be quite difficult to construct such a continuum that is tree-like but not chainable is indicated by Theorem 4. (However, Anderson and Choquet [1] have constructed a tree-like continuum which contains no chainable continuum).

THEOREM 4. *Suppose that j is a positive integer and Π is a collection of finite graphs no one of which contains more than j points of order greater than 2. Then each non-degenerate Π-like continuum contains a chainable continuum.*

Proof. Suppose that M is a non-degenerate Π-like continuum. For each positive integer n, let H_n denote a finite collection of domains of diameter less than $1/n$ covering M and containing a subcollection G_n of j domains such that each non-degenerate coherent subcollection of H_n-G_n is a simple chain (e.g. such that the nerve of H_n is an element of Π); and let $g_{n1}, g_{n2}, \ldots, g_{nj}$ denote the elements of G_n. There exists an increasing sequence $k_1, k_2, \ldots$ of positive integers such that, for each $i (1 \leqslant i \leqslant j)$, the sequence $g_{k_1 i}, g_{k_2 i}, \ldots$ has a single sequential limiting point, p_i. Let D be a domain containing all of the points $p_1, p_2, \ldots, p_j$ such that $M-D$ has a nondegenerate component K. Let ε be a positive number. There is a positive integer $r > 1/\varepsilon$ such that every element of G_{k_r} is a subset of D and, hence, the collection of all elements of H_{k_r} which intersect K is an ε-chain covering K. Thus K is a chainable subcontinuum of M.

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