COLLOQUIUM MATHEMATICUM

VOL. XXX 1974 FASC. 1

ON LEXICOGRAPHIC PRODUCTS

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The following theorem solves a problem proposed to the author (in private correspondence) by R. Telgársky:

THEOREM 1. *Let* $(X, <)$ *and* $(Y, <)$ *have paracompact order topologies. Then the set* $X \times Y$ *ordered lexicographically (i.e., according to first differences) has paracompact order topology.*

V. V. Fedorčuk had earlier remarked to R. Telgársky that if the order topology of X is compact and that of Y is paracompact, then the lexicographic order topology of $X \times Y$ is paracompact, but there is no published proof or announcement of this. It is important to note that Fedorčuk gives in [2], p. 69, an example of a paracompact order topology X such that the usual topological product of X with the space of irrationals (a paracompact ordered space) is not paracompact — in marked contrast to the result in Theorem 1.

Our proof uses a characterization of paracompactness for ordered spaces due to Gillman and Henriksen [3] which we recall here.

Definitions. Let $(X, <)$ be a (totally) ordered set and let $S, S' \subseteq X$. We say that S' is *right-cofinal* (*left-cofinal*) with S if, for each s in S, there is an s' in S' such that $s \leqslant s'$ $(s' \leqslant s)$ and, for each s' in S', there is an s in S such that $s' \leqslant s$ $(s \leqslant s')$.

By a *gap* in X we mean a pair of disjoint sets (S, T) whose union is X and such that

(i) $s < t$ for every s in S and t in T,

(ii) S has no greatest element and T has no smallest one.

Thus S does not have a supremum nor T an infimum.

Let x be a monotone sequence $\langle x_a \colon a < \lambda\rangle$ of points of X. We shall say that the sequence x *defines a gap* if either x is increasing and $\big(\{x\colon (\exists a < \lambda)\ x < x_a\}, \{x\colon (\forall a < \lambda)\ x_a < x\}\big)$ is a gap in X or x is decreasing and $\big(\{x\colon (\forall a < \lambda)\ x < x_a\}, \{x\colon (\exists a < \lambda)\ x_a < x\}\big)$ is a gap.

A *Q-sequence* is a monotone sequence $\langle x_a\colon a < \tau\rangle$ with τ a regular cardinal such that, for each limit ordinal λ not exceeding τ, the sequence $\langle x_a\colon a < \lambda\rangle$ defines a gap.

A gap (S, T) in X will be termed a *Q-gap* if there is an increasing Q-sequence right-cofinal with S (provided $S \neq \emptyset$) and a decreasing Q-sequence which is left-cofinal with T (provided $T \neq \emptyset$).

PROPOSITION. *$(X, <)$ has a paracompact order topology if and only if every gap in X is a Q-gap.*

This is the characterization in Gillman and Henriksen [3]. We now prepare for the proof of Theorem 1 by establishing two lemmas.

In what follows the letter τ is reserved for regular cardinals, and λ for limit ordinals. We adhere to the convention that $\langle x, y\rangle$ denotes an ordered pair, while (x, y) denotes an interval in an ordering dictated by context. We put $Z = X \times Y$.

LEMMA 1. *Let $\langle x_a\colon a < \tau\rangle$ be a (strictly) decreasing sequence in X and suppose that Y has a least element 0, but no greatest element. Then $\langle\langle x_a, 0\rangle\colon a < \tau\rangle$ is a Q-sequence in Z.*

Proof. Our claim is that, for any $\lambda \leqslant \tau$, $\inf\limits_{a<\lambda} \langle x_a, 0\rangle$ does not exist in Z. For suppose $\langle x, y\rangle$ is the infimum. Choose a y' in Y with $y < y'$. Then, for some $a < \lambda$, we should have

$$\langle x, y\rangle \leqslant \langle x_a, 0\rangle < \langle x, y'\rangle.$$

Hence $x = x_a$, but $\langle x, y\rangle \leqslant \langle x_{a+1}, 0\rangle < \langle x_a, 0\rangle$, which is a contradiction.

LEMMA 2. *Let $\langle x_a\colon a < \tau\rangle$ be a Q-sequence in X and suppose that Y has a least element 0. Then $\langle\langle x_a, 0\rangle\colon a < \tau\rangle$ is a Q-sequence in Z.*

Proof. First, suppose the sequence $\langle x_a\colon a < \tau\rangle$ is decreasing. If Y has no greatest element, our claim follows from Lemma 1. We assume, accordingly, that Y has a greatest element 1 and, for some $\lambda \leqslant \tau$ and some x in X, that

$$\langle x, 1\rangle = \inf_{a<\lambda} \langle x_a, 0\rangle$$

(clearly, no point $\langle x, y\rangle$ with $y < 1$ may be the infimum). But then it is trivial to verify that

$$x = \inf_{a<\lambda} x_a,$$

which is a contradiction.

We now suppose that the sequence $\langle x_a\colon a < \lambda\rangle$ is increasing and that, for some $\lambda \leqslant \tau$ and some x in X,

$$\langle x, 0\rangle = \sup_{a<\lambda} \langle x_a, 0\rangle$$

(evidently, no point $\langle x, y\rangle$ could be the supremum if $0 < y$). But it again follows trivially that

$$x = \sup_{a<\lambda} x_a,$$

a contradiction.

Proof of Theorem 1. If Y has no greatest and no least element, then the order topology of Z is homeomorphic to the disjoint sum $\bigoplus_{x\in X} (\{x\} \times Y)$, and so Z is paracompact. We may, therefore, assume that Y has a least element 0 (if Y has a greatest element, dualize the argument below by reversing the order relation, etc.).

Let (S, T) be a gap in Z. We show that it is a Q-gap. First, assume that $S \neq \emptyset$ and find a Q-sequence in Z right-cofinal with S. There are two cases. One alternative is that, for some x in X, $S\cap(\{x\} \times Y)$ is right--cofinal with S; then $\{x\} \times Y$ contains a Q-sequence right-cofinal with S, since Y has a paracompact order topology.

We consider the other alternative, when, for each $\langle x, y\rangle$ in S, there is a point $\langle x', y'\rangle \in S$ with $x < x'$. Put

$$S_x = \{x\colon \exists y \ \langle x, y\rangle \in S\}.$$

Then $S_x \times \{0\}$ is right-cofinal with S. For suppose that $\langle x, y\rangle \in S$. Choose $\langle x', y'\rangle \in S$ with $x < x'$; then $x' \in S_x$ and

$$\langle x, 0\rangle \leqslant \langle x, y\rangle < \langle x', 0\rangle \leqslant \langle x', y'\rangle .$$

Notice that S_x does not have a supremum in X, otherwise $\langle \sup S_x, 0\rangle$ would be a supremum for S. Let $\langle x_a\colon a < \tau\rangle$ be a Q-sequence in X right--cofinal with S_x. By Lemma 2, $\langle\langle x, 0\rangle\colon a < \tau\rangle$ is a Q-sequence in Z and is right-cofinal with $S_x \times \{0\}$. It is, therefore, right-cofinal with S.

We now suppose that $T \neq \emptyset$ and we proceed to find a Q-sequence left-cofinal with T.

The case where, for some x in X, $T\cap(\{x\} \times Y)$ is left-cofinal with T, presents no problems. We thus turn our attention to the case where, for each $\langle x, y\rangle \in T$, there is a point $\langle x', y'\rangle$ in T with $x' < x$. Let

$$T_x = \{x\colon \exists y \ \langle x, y\rangle \in T\}.$$

Then $T_x \times \{0\}$ is left-cofinal with T. To see this note that if $\langle x, y\rangle \in T$ and $\langle x', y'\rangle$ is chosen in T so that $x' < x$, then

$$\langle x', 0\rangle \leqslant \langle x', y'\rangle < \langle x, 0\rangle \leqslant \langle x, y\rangle .$$

We must now consider two situations according as T_x has or does not have an infimum in X.

If inf T_x exists, it is impossible that Y have a largest element. For if 1 were the largest element in Y, the point $\langle \inf T_x, 1\rangle$ would be an infimum for T in Z. This established, we may take a decreasing sequence $\langle x_a\colon a<\tau\rangle$ in X which is left-cofinal with T_x and, invoking Lemma 1, form the Q-sequence $\langle\langle x_a, 0\rangle\colon a<\tau\rangle$ in Z. The latter sequence is left-cofinal in T as required.

When T_x does not have an infimum, we argue that there is a Q-sequence in X which is left-cofinal with T_x ($X\setminus T_x$ has no supremum), say $\langle x_a\colon a<\tau\rangle$. By Lemma 2, $\langle\langle x_a, 0\rangle\colon a<\tau\rangle$ is a Q-sequence in Z and it is cofinal with T.

We have thus shown that every gap in Z is a Q-gap so, by the Proposition stated earlier, Z has a paracompact order topology.

COROLLARY. *The Sorgenfrey topology of $(X, <)$ is paracompact provided that the order topology of X is paracompact.*

Proof. The *Sorgenfrey topology* of $(X, <)$ is that generated by intervals of the form $(x_1, x_2]$ with $x_1 < x_2$. Now, the space of real numbers of the interval $[0, 1)$ with the usual (order!) topology is paracompact, whence $X\times[0, 1)$ has a paracompact lexicographic order topology. The set $X\times\{0\}$ is closed in this space, and so is a paracompact space. But the subspace topology on $X\times\{0\}$ is trivially homeomorphic to the Sorgenfrey topology of $(X, <)$.

The argument which established Theorem 1, in fact, proves a little more:

THEOREM 2. *Let $(X, <)$ have a paracompact order topology and suppose that, for each x in X, $(Y_x, <)$ has a paracompact order topology. Then the set*

$$Z = \bigcup_{x\in X}\{x\}\times Y_x,$$

ordered lexicographically, has a paracompact order topology in the following cases:

(i) *each set Y_x has a smallest and a largest element;*

(ii) *each set Y_x has a smallest (largest) element but no largest (smallest);*

(iii) *each set Y_x has no smallest and no largest element.*

Remark. The argument breaks down if the sets Y_x differ with regard to the existence of smallest and largest points and, in fact, the theorem fails to hold. For example, take $(X, <)$ to be $(\omega_1+1, <)$, so that X has a compact order topology, and $Y_a = \{a\}$ for each $a<\omega_1$, while Y_{ω_1} is to have a regressive order type ω. Then

$$Z = \bigcup_{a\leqslant\omega_1}\{a\}\times Y_a$$

contains as a closed subspace the set $\{\langle a, a\rangle\colon a<\omega_1\}$ whose subspace topology is homeomorphic to the order topology of $\langle\omega_1, <\rangle$. The latter, however, is not paracompact (see, for instance, Engelking [1], p. 228), so Z is not paracompact.

Acknowledgement. The author thanks Professor C. A. Rogers, his supervisor, for encouragement in the preparation of this note. He also acknowledges the financial support of the Science Research Council.

REFERENCES

[1] R. Engelking, *Outline of general topology*, Amsterdam - Warszawa 1968.
[2] В. В. Федорчук, *Упорядоченные множества и произведение топологических пространств*, Вестник Московского университета, Серия I, математика, механика, 21 (4), (1966), p. 66-71 (with an English summary).
[3] L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Transactions of the American Mathematical Society 77 (1954), p. 340-362.

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Reçu par la Rédaction le 7. 2. 1973