COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 2

ON A PROBLEM OF LELEK CONCERNING OPEN MAPPINGS

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1. Introduction. In 1942, Whyburn [7] proved that if f is an open mapping of a compactum X onto a compactum Y, K is any connected subset of Y, and Q any quasi-component of $f^{-1}(K)$, then $f(Q) = K$. Later on, Professor A. Lelek raised the following problem:

If f is an open mapping of a compactum X onto a compactum Y, K is any connected subset of Y, and C any component of $f^{-1}(K)$, is it true that $f(C) = K$?

Epps, Jr., working on this problem, proved that Lelek's conjecture is true under the additional assumption that X is a hereditarily locally connected continuum (see [1] and [5], Theorem 4.1).

We say that a mapping f of a topological space X onto a topological space Y is *strongly confluent* provided, for each connected subset K of Y, and for any component C of $f^{-1}(K)$, we have $f(C) = K$ (see [1]). We also say that f is *H-confluent* provided, for each subset Z of Y, each $z \in Z$, and each $x \in f^{-1}(z)$, we have

$$f[Q(f^{-1}(Z), x)] = Q(Z, z).$$

In this paper* we provide a solution to Lelek's problem, proving that open mappings on compacta are strongly confluent under the assumption that the image space Y is hereditarily locally connected (i.e., each connected subset is locally connected) (see Theorem 1). We also prove that, without this assumption, open mappings are not always strongly confluent (see Example 1).

The author wishes to thank Professor A. Lelek for several interesting conversations and the valuable suggestions.

2. Two theorems on open mappings (1). Let X be a topological space, $A \subset X$, and $x \in A$. We denote by $Q(A, x)$ the quasi-component of x

* This work is a part of Chapter II of the author's doctoral dissertation which was prepared under the direction of Professor A. Lelek.

(1) These theorems were announced in the Notices of the American Mathematical Society 23 (1976), 76T-G87.

in A. Write $Q^0(X, x) = X$ and using transfinite induction define $Q^\alpha(X, x)$ for each ordinal α as follows:

$$Q^{\alpha+1}(X, x) = Q[Q^\alpha(X, x), x] \quad \text{and} \quad Q^\lambda(X, x) = \bigcap_{\alpha<\lambda} Q^\alpha(X, x)$$

for each limit ordinal λ.

The set $Q^\alpha(X, x)$ contains x and is closed in X. We call $Q^\alpha(X, x)$ the *quasi-component of order* α. Thus the quasi-components are quasi-components of order 1. We can observe that if X is connected, then all quasi-components of each order of X are equal to X. Conversely, if $Q^1(X, x) = X$ for any $x \in X$, then X is connected.

The ordinal

$$\mathrm{nc}(X, x) = \min\{\alpha: Q^\alpha(X, x) = Q^\beta(X, x) \text{ for } \alpha < \beta\}$$

is called the *non-connectivity index of the spece X at the point x*, and it is not difficult to check that if $\gamma = \mathrm{nc}(X, x)$, then $Q^\gamma(X, x)$ is the component of the point x in X.

There are examples of spaces whose non-connectivity index is arbitrarily high. If, however, X is separable metric, then $\mathrm{nc}(X, x)$ is a countable ordinal (see [3], § 24, II, Theorem 2).

We say that a topological space is *hereditarily locally connected* (h.l.c.) provided every connected subspace of it is locally connected. Let X be a regular space. We say that X has *small inductive dimension equal to zero at the point* $p \in X$ provided, for any neighborhood V of x, there exists a closed-open neighborhood U of x in X such that $x \in U \subset V$. We write $\mathrm{ind}_p X = 0$. Finally, we say that a mapping $f\colon X \to Y$ is *0-dimensional* provided, for each $y \in Y$, $f^{-1}(y)$ has small inductive dimension equal to zero at each point of $f^{-1}(y)$.

We first prove two lemmas that we need for the proofs of the theorems.

LEMMA 1. *If X is a regular space, and A is a compact subset of X such that* $\mathrm{ind}_p A = 0$, *then for any open neighborhood G of p in X there exists an open neighborhood V of p in X such that*

$$p \in V \subset \overline{V} \subset G \quad \textit{and} \quad (\overline{V} \setminus V) \cap A = \emptyset.$$

Proof. Since $\mathrm{ind}_p A = 0$, there exists a closed-open set C, being a subset of A, such that $p \in C \subset G$. Then C and $A \setminus C$ are compact subsets of A and for each point $x \in C$ we have $x \in G \cap [X \setminus (A \setminus C)]$, which is an open subset of X.

From the regularity of X we infer that for each $x \in C$ there exists an open subset of X, say V_x, such that

$$x \in V_x \subset \overline{V}_x \subset G \cap [X \setminus (A \setminus C)]. \tag{1}$$

By compactness of C, we can find finitely many points $x_1, x_2, \ldots, x_k$ of C such that $C \subset V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_k}$. Put

$$V = V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_k}.$$

Then

$$(2) \qquad p \in C \subset V \quad \text{and} \quad \overline{V} = \overline{V}_{x_1} \cup \overline{V}_{x_2} \cup \ldots \cup \overline{V}_{x_k} \subset G.$$

Now, since $C \subset V$, we get

$$(3) \qquad C \cap (\overline{V} \setminus V) = \emptyset.$$

By (1) and (2), we derive

$$(4) \quad (A \setminus C) \cap (\overline{V} \setminus V) \subset (A \setminus C) \cap \overline{V} \\ = (A \setminus C) \cap (\overline{V}_{x_1} \cup \overline{V}_{x_2} \cup \ldots \cup \overline{V}_{x_k}) \subset (A \setminus C) \cap [X \setminus (A \setminus C)] = \emptyset.$$

By (3) and (4), finally, we obtain

$$A \cap (\overline{V} \setminus V) = [C \cap (\overline{V} \setminus V)] \cup [(A \setminus C) \cap (\overline{V} \setminus V)] = \emptyset.$$

LEMMA 2. *Let $f\colon X \to Y$ be a perfect mapping (i.e., closed with compact preimages of points) of a regular space X onto a topological space Y and let $\mathfrak{F}$ be a non-empty collection of closed non-empty subsets of X such that the following conditions are satisfied:*

(i) $\operatorname{ind} f^{-1}(y) = 0$ *for each* $y \in Y$;

(ii) *the space Y is connected and locally connected;*

(iii) *the conditions $F_1, F_2, \ldots, F_n \in \mathfrak{F}$ imply the existence of a set $F_0 \in \mathfrak{F}$ such that $F_0 \subset F_1 \cap F_2 \cap \ldots \cap F_n$;*

(iv) *the mapping $f|F\colon F \to Y$ is open for each $F \in \mathfrak{F}$.*

Then $f|\bigcap\mathfrak{F}\colon \bigcap\mathfrak{F} \to Y$ is a perfect and open mapping of $\bigcap\mathfrak{F}$ onto Y.

Proof. Since F is closed for each $F \in \mathfrak{F}$, $\bigcap\mathfrak{F}$ is a closed subset of X, therefore, $f|\bigcap\mathfrak{F}$ is a closed mapping of $\bigcap\mathfrak{F}$ into Y. If y is in the image of $\bigcap\mathfrak{F}$ under f, then $f^{-1}(y) \cap \bigcap\mathfrak{F}$ is compact. Thus $f|\bigcap\mathfrak{F}$ is a perfect mapping. To show that $f|\bigcap\mathfrak{F}$ is open let $x_0 \in \bigcap\mathfrak{F}$ and $y_0 = f(x_0)$. Let U be an open neighborhood of x_0 in $\bigcap\mathfrak{F}$. Then by (i) and Lemma 1 we derive the existence of an open neighborhood V of x_0 in X such that

$$V \cap \bigcap\mathfrak{F} \subset \overline{V} \cap \bigcap\mathfrak{F} \subset U \quad \text{and} \quad (\overline{V} \setminus V) \cap f^{-1}(y_0) = \emptyset.$$

Since y_0 is not in $f(\overline{V} \setminus V)$, we obtain $y_0 \in Y \setminus f(\overline{V} \setminus V)$ which is an open subset of the locally connected space Y. So there exists an open connected neighborhood N of y_0 in $Y \setminus f(\overline{V} \setminus V)$ such that

$$(5) \qquad y_0 \in N \subset Y \setminus f(\overline{V} \setminus V).$$

Now, let $F \in \mathfrak{F}$; since $f|F$ is a closed mapping, we get

$$(6) \qquad \overline{f(V \cap F)} \setminus f(V \cap F) \subset f(\overline{V \cap F} \setminus V \cap F) \subset f(\overline{V} \setminus V).$$

From (5) and (6) we obtain

$$N \cap [\overline{f(V \cap F)} \setminus f(V \cap F)] = \emptyset. \tag{7}$$

Since $x_0 \in \bigcap \mathfrak{F}$, we get $x_0 \in F$ for each $F \in \mathfrak{F}$, which together with $x_0 \in V$ gives $x_0 \in V \cap F$, so that

$$y_0 = f(x_0) \in f(V \cap F) \qquad (F \in \mathfrak{F}). \tag{8}$$

Now, from the connectedness of N, (7), (8), and (iv), we obtain

$$N \subset f(V \cap F) \qquad (F \in \mathfrak{F}). \tag{9}$$

We will show that $N \subset f(U)$. Let $y \in N$. Then for each $F \in \mathfrak{F}$ we have, by using (9),

$$\emptyset \neq f^{-1}(y) \cap (V \cap F) \subset f^{-1}(y) \cap \overline{V \cap F}.$$

Consider the family $\{f^{-1}(y) \cap \overline{V \cap F}\colon F \in \mathfrak{F}\}$. This is a collection of non-empty closed subsets of the compact set $f^{-1}(y)$ having the finite intersection property. To show the latest let $F_1, F_2, \ldots, F_n$ be a finite collection of elements of $\mathfrak{F}$. Then by (iii) there exists $F_0 \in \mathfrak{F}$ such that

$$F_0 \subset F_1 \cap F_2 \cap \ldots \cap F_n.$$

Therefore, by (9) we have

$$\bigcap_{i=1}^{n} [f^{-1}(y) \cap \overline{V \cap F_i}] = f^{-1}(y) \cap \bigcap_{i=1}^{n} \overline{V \cap F_i} \supset f^{-1}(y) \cap V \cap \bigcap_{i=1}^{n} F_i$$
$$\supset f^{-1}(y) \cap V \cap F_0 \neq \emptyset.$$

Thus,

$$f^{-1}(y) \cap \bigcap_{F \in \mathfrak{F}} \overline{V \cap F} \neq \emptyset.$$

We also have

$$f^{-1}(y) \cap \bigcap_{F \in \mathfrak{F}} \overline{V \cap F} \subset f^{-1}(y) \cap \overline{V} \cap \bigcap_{F \in \mathfrak{F}} F \subset f^{-1}(y) \cap U.$$

Thus, $f^{-1}(y) \cap U \neq \emptyset$, which implies $N \subset f(U)$. Hence, $f(U)$ is open in Y and the proof of Lemma 2 is complete.

THEOREM 1. *If f is an open, perfect and 0-dimensional mapping of a regular space X onto an h.l.c. space Y, $K \subset Y$ is a connected set, and $Q = Q^a[f^{-1}(K)]$ is a quasi-component of order a of $f^{-1}(K)$ (where a is any ordinal number), then $f(Q) = K$ and the mapping $f|Q$ is open.*

Proof. Since $f^{-1}(K)$ is an inverse set, $f|f^{-1}(K)$ is an open, perfect and 0-dimensional mapping of $f^{-1}(K)$ onto K, which is a locally connected subspace of Y. To prove the theorem we use transfinite induction on a. Take an arbitrary $x \in X$. If $a = 0$, then the theorem is true since

$Q^0[f^{-1}(K), x] = f^{-1}(K)$. Suppose that the theorem is true for each ordinal $\beta < \alpha$. To show that $f|Q^\alpha$ is an open mapping and $f(Q^\alpha) = K$ (where $Q^\alpha = Q^\alpha[f^{-1}(K), x]$), we distinguish two cases:

(i) $\alpha = \gamma + 1$. Then

$$Q^\alpha[f^{-1}(K), x] = Q^{\gamma+1}[f^{-1}(K), x] = Q\big[Q^\gamma[f^{-1}(K), x]\big],$$

so $Q^\alpha[f^{-1}(K), x]$ is the intersection of all closed-open subsets of $Q^\gamma[f^{-1}(K), x]$ containing x. Denote by $\mathfrak{F}$ the collection of all closed-open subsets of $Q^\gamma[f^{-1}(K), x]$ containing x. It is easy to check that all the conditions of Lemma 2 are satisfied so that $f|Q^\alpha = f|\bigcap\mathfrak{F}$ is open and $f(Q^\alpha) = f(\bigcap\mathfrak{F}) = K$.

(ii) α is a limit ordinal. Then

$$Q^\alpha = \bigcap_{\beta<\alpha} Q^\beta.$$

So if we put $\mathfrak{F} = \{Q^\beta\colon \beta < \alpha\}$, then, by Lemma 2, $f|Q^\alpha$ is open and $f(Q^\alpha) = K$.

COROLLARY 1. *If f is an open, perfect and 0-dimensional mapping of a regular space X onto an h.l.c. space Y, $K \subset Y$ is a connected set, and C is a component of $f^{-1}(K)$, then $f(C) = K$ and the mapping $f|C$ is open.*

Proof. There exists an ordinal γ such that $C = Q^\gamma[f^{-1}(K)]$. Then, by Theorem 1, $f(C) = K$ and the mapping $f|C$ is open.

THEOREM 2. *If f is a quasi-interior mapping of a compactum X onto an h.l.c. compactum Y, $K \subset Y$ is a connected set, and C is a component of $f^{-1}(K)$, then $f(C) = K$, i.e., f is strongly confluent.*

Proof. Since f is quasi-interior, it can be written as $f = ml$, where m is a monotone mapping and l is an open, perfect and 0-dimensional mapping (see [4], Corollary 3.1). Then it is immediate that m is strongly confluent and, by Corollary 1, so is the mapping l. It is easy now to check that the composition of two strongly confluent mappings is strongly confluent. Therefore, f is strongly confluent.

The following corollary generalizes an earlier result (see [1] and [5]).

COROLLARY 2. *Open mappings of compacta onto h.l.c. compacta are strongly confluent.*

3. An example of an open mapping. In this section we prove that the assumption in Theorems 1 and 2 that the image space is h.l.c. is essential. In particular, we show that an open mapping need not be strongly confluent.

Example 1. *There exists an open 0-dimensional mapping of a metric continuum onto a non-h.l.c. continuum, which is not strongly confluent.*

Proof. We begin with constructing a preliminary compactum X', the continuum X being a quotient of X'. The continuum X will be the domain of the mapping f.

Let

$$D = \{0\} \cup \left\{\frac{1}{n} : n = 1, 2, \ldots\right\} \cup \left\{1 + \frac{1}{n} : n = 1, 2, \ldots\right\}$$

and let C be the Cantor ternary set on the interval $[0, 1]$. Consider the decompositions

$$C = A_0^n \cup A_1^n \cup A_2^n$$

of the Cantor set, where for $n = 1, 2, \ldots$

$$A_0^n = C \cap \left\{x\colon 0 \leqslant x \leqslant \frac{1}{3^{n-1}}\right\}, \quad A_1^n = C \cap \left\{x\colon \frac{2}{3^n} \leqslant x \leqslant \frac{1}{3^{n-1}}\right\},$$

$$A_2^n = C \cap \left\{x\colon \frac{2}{3^n} \leqslant x \leqslant 1\right\}.$$

Now consider the subset of the xy-plane and the decomposition

$$C \times D = (C' \cup C'') \cup \bigcup_{n=1}^{\infty} C_n,$$

where

$$C' = \{(0, 0)\} \cup \left\{\left(0, \frac{1}{i}\right) : i = 1, 2, \ldots\right\}, \quad C'' = C \times \{0\},$$

$$C_1 = A_0^1 \times \{2\} \cup \bigcup_{i=1}^{\infty} A_1^1 \times \left\{1 + \frac{1}{i}\right\} \cup A_2^1 \times \{1\},$$

$$C_n = A_0^n \times \left\{1 + \frac{1}{n}\right\} \cup \bigcup_{i=n}^{\infty} A_1^n \times \left\{1 + \frac{1}{i}\right\} \cup \bigcup_{i=1}^{n-1} A_1^n \times \left\{\frac{1}{i}\right\} \cup A_2^n \times \left\{\frac{1}{n}\right\}$$
$$(n = 2, 3, \ldots).$$

This decomposition of $C \times D$ is upper-semicontinuous and satisfies the following conditions:

(i) C_n is a closed-open subset of $C \times D$ $(n = 1, 2, \ldots)$;

(ii) $C'' = p_1(C_n)$, where p_1 is the projection of $C \times D$ onto C $(n = 1, 2, \ldots)$;

(iii) $C' \cap C_n = \emptyset = C'' \cap C_n$ and $C_n \cap C_m = \emptyset$ $(n, m = 1, 2, \ldots; n \neq m)$;

(iv) $C' \cup C'' = \operatorname{Lim}_{n \to \infty} C_n$.

Now, consider the product $(C \times D) \times I$, where $I = [0, 1]$, and the equivalence relation R' in $(C \times D) \times I$ given by

$$R' = \bigcup_{n=1}^{\infty} \left\{\left((x, y, 1), (x', y', 1)\right) : (x, y), (x', y') \in C_n\right\} \cup$$
$$\cup \left\{\left((x, y, 1), (x', y', 1)\right) : (x, y), (x', y') \in C' \cup C''\right\}.$$

Let X' be the quotient space $[(C\times D)\times I]/R'$. Then the resulting space X' is the union of the cones over the sets $C'\cup C''$ and C_n $(n=1,2,\ldots)$. For simplicity, denote by a_0 the vertex of the cone over $C'\cup C''$, and by a_n the vertex of the cone over C_n $(n=1,2,\ldots)$. Then

$$a_0=\lim_{n\to\infty}a_n$$

and X' is a compact metric space with topology inherited from the Euclidean space $\boldsymbol{R}^3$. (The space X will be defined later.)

The image space Y will be the cone over the Cantor set $C\equiv C''$ with vertex a_0.

Next, we define an open mapping f' of X' onto Y as follows: If $p_1\colon C\times D\to C$ is the projection onto C (which is homeomorphic to C''), let

$$p_1\times\mathrm{id}\colon (C\times D)\times I\to C\times I$$

be the product of p_1 and the identity of I onto I. Then $p_1\times\mathrm{id}$ is an open mapping as the product of two open mappings. We define a mapping $f'\colon X'\to Y$ of X' onto Y by

$$f'(p)=\begin{cases}(p_1\times\mathrm{id})(p) & \text{if } p\neq a_i\ (i=0,1,2,\ldots),\\ a_0 & \text{if } p=a_i\ (i=0,1,2,\ldots).\end{cases}$$

To show that f' is an open mapping it suffices to prove that if G is an open subset of X', then $f'(G)$ is an open subset of Y. We distinguish two cases for the set G.

Case 1. G does not contain any vertex $a_0,a_1,a_2,\ldots$

Then $f'(G)=(p_1\times\mathrm{id})(G)$ and, therefore, $f'(G)$ is open since $p_1\times\mathrm{id}$ is an open mapping.

Case 2. G is an ε-ball around a_k for some $k\in\{1,2,\ldots\}$.

Then G is the union of rays starting from a vertex in G (it is possible that G contains more than one vertex) each one of diameter less than or equal to ε. From conditions (i) and (ii) we infer that since $p_1(C_k)=C''$, we have

$$(p_1\times\mathrm{id})(C_k\times I)=C''\times I.$$

Therefore, f' maps the cone over C_k onto Y. Consequently, there exists a $\delta>0$ such that $f'(G)$ consists of rays starting from $f'(a_k)=a_0$ and has diameter less than or equal to δ, and each ray in Y starting from a_0 has an interval of length at most δ in $f'(G)$ (one end-point of all these segments will be at a_0). Thus, $f'(G)$ is an open subset of Y.

Now, we are ready to define the continuum X and the open mapping f of X onto Y. Let R be the equivalence relation in X' given by

$$R=\big((f')^{-1}[(1,0,0)]\times(f')^{-1}[(1,0,0)]\big)\cup\{(p,p)\colon p\in X'\}.$$

Then $X = X'/R$ is a continuum. Let φ be the natural projection of X' onto X. Define a mapping $f\colon X \to Y$ by $f(x) = f'(x')$, where $x' \in X'$ is such that $\varphi(x') = x$. It is easy to check that f is a well-defined mapping of X onto Y such that $f' = f\varphi$. But the class of open mappings has the composition factor property (see [6], 5.15), and since f' is open, we conclude that f is open. The map f is 0-dimensional, since $f^{-1}(y)$ is countable for each $y \in Y$.

Finally, we prove that f is not strongly confluent. For this let B be the Knaster-Kuratowski biconnected set (see [2], p. 241) in Y. Put $K = B\setminus\{(1, 0, 0)\}$. Since B is a connected subset of Y and the point $(1, 0, 0)$ is different from the dispersion point a_0 of B, we infer that K is a connected subset of Y. Consider the preimage $f^{-1}(K)$. It consists of countably many non-degenerate connected quasi-components each one lying on the cone over some C_n $(n = 1, 2, \ldots)$ and of a non-connected quasi-component Q_0 lying on the cone over $C' \cup C''$. The set Q_0 has a non-degenerate component on the cone over C'', which is mapped onto K. It also has countably many degenerate components lying on the cone over C'. None of these components is mapped onto K. Thus, f is not strongly confluent.

Remarks. 1. Example 1 solves in the negative the problem raised by Professor A. Lelek, which was mentioned in the introduction of this paper.

2. In a discussion with the author, Professor A. Lelek asked the following question:

Let $f\colon X \to Y$ be a mapping such that, for each subset Z of Y, each point z of Z, and each point $x \in f^{-1}(z)$, we have $f[Q^a(f^{-1}(Z), x)] = Q^a(Z, z)$ for each ordinal a. Do H-confluent mappings have this property?

It is easy to check that mappings with the above-mentioned property are H-confluent and strongly confluent, but the converse is not true. In Example 1 we consider an open, hence H-confluent mapping which does not have this property. Namely, if Z is the connected set K of Example 1, then $f^{-1}(Z)$ has countably many quasi-components of order 2, which are degenerate.

3. The conclusion in Theorem 1 that the restriction of the mapping f to a quasi-component Q of the preimage of any connected set K is also open may fail without the assumption that the image space is h.l.c. To prove this let K be the connected set of Example 1 and let Q_0 be the quasi-component of $f^{-1}(K)$ lying on the cone over $C' \cup C''$. Then the restriction of f on Q_0 is not even a quasi-interior mapping of Q_0 onto K. To see this let y be a point of K lying on $\{(0, 0)\} \times I$ and different from the vertex a_0. Then $f^{-1}(y)$ has countably many degenerate components. Let D be one of these components lying on the set $\{(0, 1)\} \times I$. Let U be

an open neighborhood of D in Q_0 lying entirely on the cone over C'. Then, clearly, $f(U)$ is a subset of $\{(0, 0)\} \times I$, so $y \notin \operatorname{Int} f(U)$.

4. The following problem can be raised:

Let $f\colon X \to Y$ be an open mapping of a locally connected compactum X onto a non-h.l.c. compactum Y. Is f strongly confluent?

Professor A. Lelek has constructed an open, light mapping from a locally connected continuum onto a continuum which is not strongly confluent, thus solving the problem in the negative.

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Reçu par la Rédaction le 28. 10. 1976