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A NOTE ON NUMBERS WITH GOOD FACTORIZATION PROPERTIES

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In this note we give a very simple proof of the following result:

THEOREM. *Let K be an algebraic number field and h its class number. Let $F_k(x)$ be the number of natural numbers $n \leqslant x$ which have at most k distinct factorizations into irreducibles in K, and let $G_k(x)$ be the number of such $n \leqslant x$ which have at most k factorizations of different lengths.*

If $h \geqslant 2$, then $F_k(x) = o(x)$, and if $h \geqslant 3$, then also $G_k(x) = o(x)$ (1).

Proof. The following lemma is an immediate corollary of the Erdös-Kac theorem (see e.g. [1]) but it can be also proved easily along the lines of [5]:

LEMMA 1. *Let P be a set of primes with $\sum_{p \in P} p^{-1}$ divergent, and let m be fixed. Then almost all n's have at least m distinct factors from P.*

Now let $X_1, \ldots, X_{h-1}$ be the non-principal ideal classes of K and let P_i for $i = 1, 2, \ldots, h-1$ be the set of those rational primes which have at least one prime ideal factor in X_i. Write $\omega_i(n)$ for the number of distinct prime factors of n belonging to P_i.

The following results are proved in [4] (see lemma 1) and [3] (part I, p. 66 and 67):

LEMMA 2. (i) *If $h > 1$ and n has at most k factorizations in K, then there is a constant $B = B(K, k)$ with $\omega_i(n) \leqslant B$ $(i = 1, \ldots, h-1)$.*

(ii) *If $h(K) > 2$ and n has at most k factorizations in K, then*

$$\min\{\omega_1(n), \ldots, \omega_{h-1}(n)\} \leqslant C$$

with some constant $C = C(k, K)$.

LEMMA 3. *For $i = 1, 2, \ldots, h-1$, the series $\sum_{p \in P_i} p^{-1}$ is divergent.*

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(1) For non-normal K this result is new. For normal K the evaluation $F_k(x)$. $G_k(x) = O(x(\log x)^{-A})$ $(A > 0)$ was obtained in [3] using a non-elementary approach,

Proof. For $i = 1, \ldots, h-1$ and $\operatorname{Re} s > 1$ we have (cf. [2], p. 33)

$$\sum_{\substack{\mathfrak{p} \in X_i,\\ f(\mathfrak{p})=1}} N(\mathfrak{p})^{-s} = \frac{1}{h(K)} \log \frac{1}{s-1} + g(s),$$

where $f(\mathfrak{p})$ is the degree of $\mathfrak{p}$ and $g(s)$ is regular for $\operatorname{Re} s \geqslant 1$. Let, for any p prime, $a_i(p)$ be the number of prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) = p$. Then, for real $s > 1$, one has

$$\sum_{\substack{\mathfrak{p} \in X_i,\\ f(\mathfrak{p})=1}} N(\mathfrak{p})^{-s} = \sum_{p \in P_i} a_i(p) p^{-s} \leqslant n \sum_{p \in P_i} p^{-s}$$

(with n being the degree of K), so the series $\sum_{p \in P_i} p^{-1}$ has to diverge.

Now, by the lemmas, we get readily

$$F_k(x) \leqslant \mathcal{N}\{n \leqslant x:\ \omega_1(n) \leqslant B\} = o(x)$$

and

$$G(x) \leqslant \sum_i \mathcal{N}\{n \leqslant x:\ \omega_i(n) \leqslant C\} = o(x)$$

as asserted.

Note, however, that this method does not permit to obtain evaluations of the order $O(x/\log^a x)$ with $a > 0$ (**P 843**).

REFERENCES

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[3] W. Narkiewicz, *On algebraic number fields with non-unique factorization*, Colloquium Mathematicum 12 (1964), p. 60-68; II, ibidem 14 (1966), p. 49-58.

[4] — *Numbers with unique factorization in an algebraic number field*, Acta Arithmetica 21 (1972), p. 313-322.

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