COLLOQUIUM MATHEMATICUM

VOL. XVII 1967 FASC. 2

INDEPENDENCE IN SEPARABLE VARIABLES ALGEBRAS

BY

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In this note we shall mean by an *algebra* $\mathscr{A}$ a pair $\langle A; \boldsymbol{F}\rangle$, where A is a non-empty set and $\boldsymbol{F}$ is a class of *fundamental* operations. Every f from $\boldsymbol{F}$ is a function of several variables which associates with each system $x_1, x_2, \ldots, x_n$ of elements of A an element $f(x_1, x_2, \ldots, x_n) \in A$. $\boldsymbol{C}(E)$ denotes the subalgebra generated by a set $E \subset A$. By $\boldsymbol{A}(\mathscr{A})$ (or briefly $\boldsymbol{A}$) we shall denote the class of all algebraic functions, i.e. the smallest class of operations containing trivial operations

$$e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$$

and closed under compositions with the fundamental operations. The subclass of all n-ary algebraic operations will be denoted by $\boldsymbol{A}^{(n)}$, $n \geqslant 1$. Further, by $\boldsymbol{A}^{(0)}$ or $\boldsymbol{C}(\emptyset)$ we shall denote the set of all values of constant algebraic operations. Elements belonging to $\boldsymbol{A}^{(0)} = \boldsymbol{C}(\emptyset)$ will be called *algebraic constants*. We say that the elements of a set I ($I \subset A$) are *independent* if for each system of n different elements $a_1, a_2, \ldots, a_n$ from I and for each pair of operations $f, g \in \boldsymbol{A}^{(n)}$ the equation

$$f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$$

implies that f and g are identical in $\mathscr{A}$. We shall denote the class of all independent sets of an algebra $\mathscr{A}$ by Ind $(\mathscr{A})$. The above definitions in a more detailed form and theorems concerning them are given in [4] and [5].

In the present paper a condition fulfilled in separable variables algebras (condition JIS) will be defined, and its connection with exchange of independent sets property will be examined.

Definition 1. An algebra $\mathscr{A}$ is called an *algebra with separable k variables* ($k \geqslant 1$) if for every pair $f, g \in \boldsymbol{A}^{(n)}$, $n > k$, there exist functions $f_0 \in \boldsymbol{A}^{(k)}$ and $g_0 \in \boldsymbol{A}^{(n-k)}$ such that the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$$

is equivalent to the equation

$$f_0(x_1, x_2, \ldots, x_k) = g_0(x_{k+1}, x_{k+2}, \ldots, x_n).$$

An algebra with separable k variables for all $k = 1, 2, \ldots$ will be called, briefly, a *separable variables algebra*. The class of separable variables algebras will be denoted by SV.

This notion was introduced by Marczewski in [3]. Obviously, Abelian groups and linear spaces belong to SV. Generally, one can prove that a group with multiple operators is separable variables algebra if it is an Abelian group with multiple operators (the reader will find suitable definitions in [2]); hence, in particular, it follows that among rings only rings with zero-multiplication belong to SV. Other examples of separable variables algebras are given in [1], where also the representation theorem for these algebras is proved. Separable variables algebras coincide with the so-called quasi-linear algebras, as well as with algebras with separable k variables (see [1]).

In [3] it was proved that separable variables algebras have the exchange of independent sets property (EIS).

Definition 2. An algebra $\mathscr{A}$ has the *exchange of independent sets property* ($\mathscr{A} \in \mathrm{EIS}$) if for any three subsets $P, Q, R \subset A$ such that $R, P \cup Q \in \mathrm{Ind}(\mathscr{A})$, $P \cap Q \neq \emptyset$ and $R \subset \boldsymbol{C}(Q)$, we have $P \cup R \in \mathrm{Ind}(\mathscr{A})$.

For example: Abelian groups, Boolean algebras (and, more generally, the so-called Post algebras [6]), linear spaces, and v^*-algebras have EIS-property (see [3]). Moreover, one can prove that every finite algebra generated independently by one or two elements has the EIS property.

The property given below is stronger than the EIS-property.

Definition 3. An algebra $\mathscr{A}$ has the *JIS-property* (*joining independent sets property*) if for any two independent sets P and Q such that $\boldsymbol{C}(P) \cap \boldsymbol{C}(Q) = \boldsymbol{C}(P \cap Q)$ we have $P \cup Q \in \mathrm{Ind}(\mathscr{A})$.

We shall prove that this property is equivalent to the following one:

Definition 4. An algebra $\mathscr{A}$ has the *JIS*-property* if for any two independent sets P and Q such that $\boldsymbol{C}(P) \cap \boldsymbol{C}(Q) = \boldsymbol{C}(\emptyset)$ we have $P \cup Q \in \mathrm{Ind}(\mathscr{A})$.

The classes of algebras having the last defined properties will be denoted by JIS and JIS*, respectively.

Notice that the condition formulated in definition 3 is a conversion of the multiplicatibility of operation $\boldsymbol{C}$ on the subsets of the independent set proved by Marczewski (cf. [5], p. 56). Namely

(M) if $P, Q \subset I \in \mathrm{Ind}(\mathscr{A})$, then $\boldsymbol{C}(P) \cap \boldsymbol{C}(Q) = \boldsymbol{C}(P \cap Q)$.

We shall prove successively a few results on the just defined properties.

(i) JIS = JIS*.

Proof. Let $\mathscr{A} \in$ JIS and let $C(P) \cap C(Q) = C(\emptyset)$, where P and Q are independent sets of $\mathscr{A}$. From the obvious inclusion

$$C(P \cap Q) \subset C(P) \cap C(Q) \tag{1}$$

we immediately get $C(P \cap Q) = C(\emptyset) = C(P) \cap C(Q)$, which implies $P \cup Q \in \mathrm{Ind}(\mathscr{A})$ and shows that JIS implies JIS*.

Conversely, suppose that $\mathscr{A} \in$ JIS*, $P, Q \in \mathrm{Ind}(\mathscr{A})$ and $C(P \cap Q) = C(P) \cap C(Q)$. By virtue of (1) it suffices to prove that $C(P) \cap C(Q \setminus P) \subset C(\emptyset)$. Let $x \in C(P) \cap C(Q \setminus P)$. Evidently, x must belong to $C(P) \cap C(Q) = C(P \cap Q)$. Since $x \in C(Q \setminus P)$ and $P \cap Q$ and $Q \setminus P$ are disjoint subsets of the independent set Q, we infer by virtue of (M) that $x \in C(\emptyset)$. Since $\mathscr{A}$ has JIS*-property, $C(P) \cap C(Q \setminus P) = C(\emptyset)$, and P and Q are independent sets, their union $P \cup Q$ is an independent set too. Hence $\mathscr{A} \in$ JIS.

Next we shall prove that joining independent sets property is stronger than exchange of independent sets property:

(ii) JIS* $\subset$ EIS.

Proof. Suppose that $P \cup Q, R \in \mathrm{Ind}(\mathscr{A})$, $P \cap Q = \emptyset$ and $R \subset C(Q)$. From (M) and from the independence of $P \cup Q$ it follows that $C(P) \cap C(Q) = C(\emptyset)$. Since, evidently, $C(R) \subset C(Q)$, we also have $C(P) \cap C(R) = C(\emptyset)$. Taking into account that $P, R \in \mathrm{Ind}(\mathscr{A})$ and $\mathscr{A} \in$ JIS*, we have $P \cup R \in \mathrm{Ind}(\mathscr{A})$ and thus $\mathscr{A}$ has the exchange of independent sets property.

Now we shall show that separable variables algebras have joining independent sets property:

(iii) SV $\subset$ JIS*.

Proof. Assume that $P, Q \in \mathrm{Ind}(\mathscr{A})$ and $C(P) \cap C(Q) = C(\emptyset)$ (whence, by (1), $P \cap Q = \emptyset$). One needs to prove that $P \cup Q$ is an independent set. Obviously, it suffices to consider finite sets P and Q. Let $P = \{a_1, a_2, \ldots, a_m\}$ and $Q = \{b_1, b_2, \ldots, b_n\}$. Suppose that $P \cup Q \notin \mathrm{Ind}(\mathscr{A})$, i.e. that there are different algebraic operations f and g such that

$$f(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n) = g(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n). \tag{2}$$

Since $\mathscr{A}$ is an separable variables algebra, there are algebraic operations $f_0 \in A^{(m)}(\mathscr{A})$ and $g_0 \in A^{(n)}(\mathscr{A})$ such that the equation

$$f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = g(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) \tag{3}$$

is equivalent to the equation

$$f_0(x_1, x_2, \ldots, x_m) = g_0(y_1, y_2, \ldots, y_n). \tag{4}$$

Thus from (2) we obtain

$$f_0(a_1, a_2, \ldots, a_m) = g_0(b_1, b_2, \ldots, b_n). \tag{5}$$

Denoting the last element by c we have $c \in \boldsymbol{C}(P) \cap \boldsymbol{C}(Q)$, whence, in virtue of the assumption, $c \in \boldsymbol{C}(\emptyset)$. In view of independence of P and Q we infer that

$$f_0(x_1, x_2, \ldots, x_m) = g_0(y_1, y_2, \ldots, y_n) = c$$

for every $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in A$. Equation (4) is therefore always satisfied for any elements of A, and consequently equation (3) is too. This contradicts the assumption that $f \neq g$, which completes the proof.

The following theorem is a simple consequence of (i), (iii) and (M):

THEOREM. *Let $\mathscr{A}$ be a separable variables algebra and let P and Q be its independent subsets. The union $P \cup Q$ is independent if and only if* $\boldsymbol{C}(P \cap Q) = \boldsymbol{C}(P) \cap \boldsymbol{C}(Q)$.

Recapitulating, we have shown that

$$\mathrm{SV} \subset \mathrm{JIS} = \mathrm{JIS}^* \subset \mathrm{EIS}.$$

Examples of a trivial algebra and a Boolean algebra show that neither the first nor the second sign of inclusion can be replaced by that of equality.

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Reçu par la Rédaction le 18. 9. 1965

COLLOQUIUM MATHEMATICUM

VOL. XVII 1967 FASC. 2

ON A NEW NOTION OF INDEPENDENCE IN UNIVERSAL ALGEBRAS

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1. Introduction. Two different notions of independence are used in abelian group theory. The classical notion is the following: the elements $a_1, \ldots, a_k$ of an abelian group are independent if

$$\sum_{i=1}^{k} n_i a_i = 0 \text{ implies } n_1 = n_2 = \ldots = n_k = 0; \tag{1.1}$$

hence a single element a is independent if and only if it is torsion free.

In recent papers a new notion of independence (introduced by T. Szele) has frequently been used:

the elements $a_1, \ldots, a_k$ are independent if

$$\sum_{i=1}^{k} n_i a_i = 0 \text{ implies } n_1 a_1 = \ldots = n_k a_k = 0. \tag{1.2}$$

Hence a single element a is always independent (see e.g. [2]).

The first notion is connected with the notion of free abelian groups.

The notion of a free universal algebra was introduced by Birkhoff [1] and based on this Marczewski [4] gave a general notion of independence in universal algebras.

In this note an attempt will be made to generalize Marczewski's notion of independence in such a way that when applied to abelian groups it should be indentical with (1.2).

This will be achieved by defining the order of an element in a universal algebra.

The basic notions are given in § 2, the order of an element is defined in § 3 while in § 4 the new notion of independence is given. The characterization theorem of weak independence is proved in § 5. Some of its consequences and several unsolved problems are listed in § 6.

It should be noted that all the notions introduced in § 2 are standard ones and are given here only for completeness sake. However, the notion of the order of an element — however evident it is — seems to be new.

Most of the results of this paper were contained in my mimeographed note [3], which had a limited distribution in 1962.

2. Some notions and notation. An *algebra* is a couple $(A; F)$ where A is a set and F is a collection of fundamental operations. Every operation $f \in F$ is finitary, $f = f(x_1, \ldots, x_n)$ (n is an integer and depends on f), which means that if $(a_1, \ldots, a_n)$ is an n-tuple of elements of A, then $f(a_1, \ldots, a_n)$ is a well defined element of A.

Let $B \subseteq A$; we call $(B; F)$ a *subalgebra* of $(A; F)$ if $a_1, \ldots, a_n \in B$ and $f = f(x_1, \ldots, x_n) \in F$ imply $f(a_1, \ldots, a_n) \in B$.

Let $(A; F)$ and $(B; F)$ be algebras and $h\colon x \to xh$ a many-one mapping of A into B. The mapping h is called a *homomorphism* if

$$f(x_1, \ldots, x_n)h = f(x_1 h, \ldots, x_n h)$$

holds identically for every $f \in F$. Accordingly, an *isomorphism* h is a homomorphism which is one-to-one and onto ($Ah = B$); an *endomorphism* is a homomorphism of $(A; F)$ into itself, an *automorphism* is an isomorphism of $(A; F)$ with itself.

A congruence relation Θ on $(A; F)$ is an equivalence relation on A which has the substitution property:

(SP) if $a_i \equiv b_i(\Theta)$, $i = 1, 2, \ldots, n$, then $f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)(\Theta)$ for every $f \in F$.

Let A/Θ denote the set of equivalence classes modulo Θ and a/Θ ($a \in A$) the equivalence class represented by a. Then $(A/\Theta; F)$ is an algebra where for every $f \in F$ we put

$$f(a_1/\Theta, \ldots, a_n/\Theta) = f(a_1, \ldots, a_n)/\Theta.$$

The set of all congruence relations on $(A; F)$ is denoted by $C(A; F)$.

Let $\Theta_1, \Theta_2 \in C(A; F)$. We put $\Theta_1 \leqslant \Theta_2$ if $x \equiv y(\Theta_1)$ implies $x \equiv y(\Theta_2)$. This makes $C(A; F)$ a partially ordered set; it can be easily proved that the l.u.b.: $\Theta_1 \cup \Theta_2$ and g.l.b.: $\Theta_1 \cap \Theta_2$ always exist. $\mathfrak{C}(A; F) = \big(C(A; F); \cup, \cap\big)$ is a lattice, it is called the *congruence lattice* of $(A; F)$.

The class $A^{(n)}$ $(n = 1, 2, \ldots)$ of algebraic operation of n-variables is the smallest class satisfying the following two conditions:

(2.1) the trivial operations e_i^n defined by $e_i^n(x_1, \ldots, x_n) = x_i$ $(i = 1, 2, \ldots n)$ are in $A^{(n)}$;

(2.2) if $g_1, \ldots, g_k \in A^{(n)}$ and $f = f(x_1, \ldots, x_k) \in F$, then $f(g_1, \ldots, g_k) = f\big(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\big)$ is also in $A^{(n)}$.

Let $\mathscr{K}$ be a fixed class of algebras $(A; F)$.

An equivalence relation on $A^{(n)}$ is defined as follows: let $f, g \in A^{(n)}$; we write $f \equiv g$ if $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$ for every $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$.

Let $A_{\mathscr{K}}^{(n)}$ describe the equivalence classes under this equivalence relation. We can define the operations on $A_{\mathscr{K}}^{(n)}$ in a natural way; formula (2.2) shows that $(A_{\mathscr{K}}^{(n)}; F)$ is an algebra; this will be denoted by $\mathfrak{A}_{\mathscr{K}}^{(n)}$.

We put $A^{(\omega)} = A^{(0)} \cup A^{(1)} \cup A^{(2)} \cup \ldots$ and we define an equivalence: by $f \equiv g$ $(f \in A^{(k)}, g \in A^{(l)})$ it for every $a_1, a_2, \ldots, a_{\max(k,l)} \in A$, $(A; F) \in \mathscr{K}$ the equality $f(a_1, \ldots, a_k) = g(a_1, \ldots, a_l)$ holds. The equivalence classes will be denoted by $A_{\mathscr{K}}^{(\omega)}$ and the corresponding algebra $(A_{\mathscr{K}}^{(\omega)}; F)$ by $\mathfrak{A}_{\mathscr{K}}^{(\omega)}$.

Let $H \subseteq A$; we define the subset $[H]$ of A by $a \in [H]$ if there exists an integer n, and $f \in A^{(n)}$ and $h_1, \ldots, h_n \in H$ such that $f(h_1, \ldots, h_n) = a$.

Then $([H]; F)$ is a subalgebra of $(A; F)$; it is the subalgebra generated by H.

If A, B are sets, $A - B$ denotes the set theoretical difference. $\{a_1, \ldots, a_n\}$ denotes the set whose elements are $a_1, \ldots, a_n$. The notation $[\{a_1, \ldots\}]$ is replaced by $[a_1, \ldots]$.

3. The order of an element. Let $\mathscr{K}$ be a class of algebras, $(A; F) \in \mathscr{K}$, $a \in A$. The order of a is defined as follows:

Consider the mapping

$$e_1^1 \to a;$$

this has a unique extension to a homomorphism h of $\mathfrak{A}_{\mathscr{K}}^{(1)}$ into $(A; F)$; let $O(a)$ denote the congruence relation induced by h; we call $O(a)$ the *order* of a.

It is obvious that $O(a)$ is uniquely determined by a, $(A; F)$ and $\mathscr{K}$. Further, $O(a) \in C(A_{\mathscr{K}}^{(1)}; F) = C(\mathfrak{A}_{\mathscr{K}}^{(1)})$.

We first give a few examples:

3.1. Let $\mathscr{K}$ be the class of all additive groups. Then $\mathfrak{A}_{\mathscr{K}}^{(1)}$ is isomorphic to the group $\mathfrak{J}$ of integers, let $e_1^1 \to 1$ under this isomorphism. Let $\mathfrak{G} \in \mathscr{K}$, $a \in G$. Then the mapping $1 \to a$ has a unique extension to a homomorphism of $\mathfrak{J}$ into $\mathfrak{G}$. It is easy to see, $O(a)$ is the congruence modulo n, where n is the least integer with $na = 1$. This $O(a)$ is completely described if we give this n, which is usually called the order of a.

3.2. Let $\mathscr{K}$ be the class of all semi-groups. Now $\mathfrak{A}_{\mathscr{K}}^{(1)}$ is isomorphic to $\mathfrak{N}$, the (additive) semi-group of positive integers, again $e_1^1 \to 1$ under this isomorphism. In this case $O(a)$ can be described by a pair of non-negative integers (m, n) as follows: $x \equiv y \; \big(O(a)\big)$ $(x, y \in I)$ if and only if $x = y$ or $x > m$, $y > m$ and n divides $x - y$.

3.3. $\mathscr{K}$ is the class of all right modules over a ring $(R; +, \cdot)$. This may also be included in the above discussion in the usual way by making every element $r \in R$ correspond to a unary operation f_r and put F

$= \{+, f_r\}_{r \in R}$ and considering a right-module M as an algebra $(M; F)$. Then $\mathfrak{A}^{(1)}_{\mathscr{K}}$ *is isomorphic to* $(R; F)$ *and* $O(a)$ *may be identified with the* class containing the zero of $\mathfrak{R}$, which is an ideal I_a. Usually, this ideal I_a is called the order of a.

These examples show that the notion of an order of an element is a natural generalization of known concepts.

The following propositions show the usefulness of this notion:

3.4. *Let* $(A; F)$, $(B; F) \in \mathscr{K}$ *and* h *be a homomorphism of* $(A; F)$ *into* $(B; F)$, $a \in A$.

Then

$$O(a) \leqq O(ah). \tag{3.5}$$

To prove this we consider the homomorphisms:

$$h_1\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (A; F);\ e_1^1 \to a;$$

$$h_2\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (B; F);\ e_1^1 \to ah.$$

Then

$$h_1 h = h_2,$$

which implies that $x \equiv y\big(O(ah)\big)$ if and only if $xh_2 = yh_2$, i.e. if $(xh_1)h = (yh_1)h$. Thus $xh_1 = yh_1$, implies $xh_2 = yh_2$, i.e. $x \equiv y\big(O(a)\big)$ implies $x \equiv y\big(O(ah)\big)$.

A partial converse of 3.4 holds too:

3.6. *Let* $(A; F)$, $(B; F) \in \mathscr{K}$, $a \in A$, $b \in B$ *and suppose* $O(a) \leqq O(b)$. *Then there exists a homomorphism*

$$h\colon ([a]; F) \to ([b]; F),$$

carrying a *into* b $(b = ah)$.

To prove this consider the homomorphism:

$$h_1\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (A; F),\ e_1^1 \to a;$$

$$h_2\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (B; F),\ e_1^1 \to b.$$

We define h as follows: let $a_1 \in [a]$, then there exists an $a_2 \in A^{(1)}_{\mathscr{K}}$ with $a_1 = a_2 h_1$; let $a_1 h = a_2 h_2$.

First we have to prove that h is uniquely defined. Indeed, if $a_2 h_1 = a_3 h_1$ $(a_3 \in A^{(1)}_{\mathscr{K}})$, then $a_2 \equiv a_3\big(O(a)\big)$, which implies $a_2 \equiv a_3\big(O(b)\big)$, i.e. $a_2 h_2 = a_3 h_2$.

To show the substitution property let $f \in F$, $f = f(x_1, \ldots, x_n)$, take $a_1, \ldots, a_n \in [a]$; then

$$a_i = p_i(a), \quad p_i \in A^{(1)}, \quad i = 1, \ldots, n.$$

We want to show

$$f(a_1, \ldots, a_n)h = f(a_1 h, \ldots, a_n h).$$

Let

$$a_i' \in A_{\mathscr{K}}^{(1)}, \quad a_i' h_1 = a_i, \quad i = 1, \ldots, n,$$

$$c \in A_{\mathscr{K}}^{(1)}, \quad ch_1 = f(a_1, \ldots, a_n).$$

Then

$$c \equiv f(a_1', \ldots, a_n')\big(O(a)\big),$$

thus

$$c \equiv f(a_1', \ldots, a_n')\big(O(b)\big),$$

and also

$$ch_2 = f(a_1', \ldots, a_n')h_2.$$

Thus

$$f(a_1, \ldots, a_n)h = ch_2 = f(a_1', \ldots, a_n')h_2 = f(a_1' h_2, \ldots, a_n' h_2)$$
$$= f(a_1 h, \ldots, a_n h).$$

It should be noted that 3.6 is new only in this form. In fact it is a special case of the so called Second Isomorphism Theorem, which is a part of the folklore.

3.7. The order of an element (a, b), in the direct product of $(A; F)$ and $(B; F)$, can be computed as follows:

$$O(a, b) = O(a) \cap O(b), \tag{3.8}$$

if (A, F), (B, F) and $(A \times B, F)$ are in $\mathscr{K}$.

Let $p_1, p_2 \in A_{\mathscr{K}}^{(1)}$. (3.8) means that $p_1 \equiv p_2\big(O(a, b)\big)$ if and only if $p_1 \equiv p_2\big(O(a)\big)$ and $p_1 \equiv p_2\big(O(b)\big)$. Since $p_1 \equiv p_2\big(O(a, b)\big)$ means $p_1\big((a, b)\big) = p_2\big((a, b)\big)$ and so on, we get that we have to prove the following: $p_1\big((a, b)\big) = p_2\big((a, b)\big)$ if and only if $p_1(a) = p_2(a)$ and $p_1(b) = p_2(b)$, which holds by definition.

4. Independence and weak independence. Marczewski's notion of independence is defined as follows:

4.1. Let $\mathscr{K}$ be a class of algebras, $(A; F) \in \mathscr{K}$, $a_1, \ldots, a_n \in A$. We say that the sequence $a_1, \ldots, a_n$ is *independent* if

$$p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n), \quad p_1, p_2 \in A^{(n)},$$

imply

$$p_1 \equiv p_2.$$

It may be remarked that Marczewski's definition is restricted to the case when $\mathscr{K}$ consists only of $(A; F)$; some of his results, however, remain true for an arbitrary class $\mathscr{K}$. The characterization theorem of independent sequences is the following:

4.2. *Let $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$. Then the following conditions are equivalent:*

(4.2.1) *$a_1, \ldots, a_n$ is an independent sequence;*

(4.2.2) *let $b_1, \ldots, b_n \in B$, $(B; F) \in \mathscr{K}$ and p: $a_i \to b_i$, $i = 1, \ldots, n$. Then p can be extended to a homomorphism of $([a_1, \ldots, a_n]; F)$ into $(B; F)$;*

(4.2.3) *the mapping p: $e_i^n \to a_i$ can be extended to an isomorphism h of $\mathfrak{A}_{\mathscr{K}}^{(n)}$ onto $([a_1, \ldots, a_n]; F)$.*

The equivalence of (4.2.1) and (4.4.2) is stated in [4]; I am sure that Marczewski knows that they are equivalent to (4.2.3) as well, however, I cannot give a reference.

An important corollary of 4.2 (which is also due to Marczewski) is:

4.3. *If $a_1, \ldots, a_n$ is independent, then so is $a_{i_1}, \ldots, a_{i_n}$, where $j \to i_j$ is any permutation of $1, \ldots, n$.*

Thus we can speak of an independent set $a_1, \ldots, a_n$, because the ordering does not matter.

4.4. *An element a is independent if and only if a is torsion free, i.e. $O(a) = \omega$.*

This is trivial by (4.2.3) and the definition of $O(a)$.

Now we give the definition of weak independence.

4.5. Let $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$. We say that the sequence $a_1, \ldots, a_n$ is *weakly independent* if

$$p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n), \quad p_1, p_2 \in A^{(n)}, \tag{4.5.1}$$

imply

$$p_1(b_1, \ldots, b_n) = p_2(b_1, \ldots, b_n) \tag{4.5.2}$$

for every $b_1, \ldots, b_n \in B$, $(B; F) \in \mathscr{K}$, for which

$$O(a_i) \leqslant O(b_i), \quad i = 1, \ldots, n. \tag{4.5.3}$$

First, let us see some trivial consequences of this definition.

4.6. *Suppose $a_1, \ldots, a_n$ are torsion free elements. Then $a_1, \ldots, a_n$ is independent if and only if it is weakly independent.*

The difference between independence and weak independence is condition (4.5.3). However, if $O(a_1) = \ldots = O(a_n) = \omega$, then (4.5.3) is no restriction on the choice of the b_i and hence in this case the two notions are equivalent.

4.7. *If $\mathscr{K}$ is a subclass of lattices, independence and weak independence are equivalent.*

Obviously, since in a lattice every element is torsion free.

5. Characterizations of weak independence. We would like to get a result analogous to 4.2. In order to achieve that we need some notation.

The algebra $\mathfrak{A}_{\mathscr{K}}^{(n)}$ is generated by $e_1^n, \ldots, e_n^n$, and the subalgebra $\mathfrak{A}_i$ generated by e_i^n is isomorphic to $\mathfrak{A}_{\mathscr{K}}^{(1)}$. Suppose we are given n congruence relations $\Theta_1, \ldots, \Theta_n$ of $\mathfrak{A}_{\mathscr{K}}^{(1)}$. Consider Θ_i as a congruence relation on $\mathfrak{A}_i$.

Take a congruence relations Θ of $\mathfrak{A}_{\mathscr{K}}^{(n)}$ having the following properties:

(5.1) the restriction of Θ to $\mathfrak{A}_i$ is $\geqslant \Theta_i$ $(i = 1, 2, \ldots, n)$;

(5.2) $\mathfrak{A}_{\mathscr{K}}^{(n)}/\Theta$ is isomorphic to a subalgebra of an algebra in $\mathscr{K}$.

If there exists a congruence relation which is the smallest one having properties (5.1) and (5.2), then it will be denoted by $\Sigma\Theta_i$.

5.3. *Let $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$. Then the following conditions are equivalent:*

(5.3.1) *$a_1, \ldots, a_n$ is a weakly independent sequence;*

(5.3.2) *let $b_1, \ldots, b_n \in B$, $(B; F) \in \mathscr{K}$, and $O(a_i) \leqslant O(b_i)$; then the mapping p: $a_i \to b_i$ $(i = 1, \ldots, n)$ can be extended to a homomorphism of $([a_1, \ldots, a_n]; F)$ into $(B; F)$;*

(5.3.3) *$\Sigma O(a_i)$ exists and*

$$\mathfrak{A}_{\mathscr{K}}^{(n)} / \Sigma O(a_i) \cong ([a_1, \ldots, a_n]; F), \qquad e_i^n / \Sigma O(a_i) \to a_i .$$

Suppose that $a_1, \ldots, a_n$ is weakly independent and the p of (5.3.2) is given. Define h as follows:

$$q(a_1, \ldots, a_n)h = q(b_1, \ldots, b_n) \qquad \text{for every } q \in A^{(n)}.$$

Obviously, h maps $[a_1, \ldots, a_n]$ into $(B; F)$. This mapping is well-defined since $q_1(a_1, \ldots, a_n) = q_2(a_1, \ldots, a_n)$ $(q_1, q_2 \in A^{(n)})$ implies by 4.5 that $p(b_1, \ldots, b_n) = q(b_1, \ldots, b_n)$.

The mapping h is an extension of p since $a_i h = e_i^n(a_1, \ldots, a_n)h = e_i^n(b_1, \ldots, b_n) = b_i$.

Finally, h is a homomorphism. The proof of this is very similar to 3.6, so it can be omitted.

Thus (5.3.1) implies (5.3.2).

Next suppose that (5.3.2) holds and consider the mapping $e_i^n \to a_i$; this can be extended to a homomorphism h of $\mathfrak{A}_{\mathscr{K}}^{(n)}$ into $(A; F)$. Let Θ be the congruence relation induced by h. Then Θ satisfies (5.1) and (5.2). Indeed, if we restrict h to $\mathfrak{A}_i$, then we get a homomorphism of $\mathfrak{A}_i$ into $(A; F)$ carrying e_i^n into a_i. Since $\mathfrak{A}_i \cong \mathfrak{A}_{\mathscr{K}}^{(1)}$, we get that the congruence relation induced by the restriction of h on $\mathfrak{A}_i$ is $\geqq O(a_i)$. Thus (5.1) is verified; (5.2) is obvious. Now we prove that Θ is the smallest one satisfying (5.1) and (5.2). Indeed, if Φ satisfies (5.1) and (5.2), then consider $(B; F) \in \mathcal{K}$ of which $\mathfrak{A}^{(n)}_{\mathcal{K}}/\Phi$ is a subalgebra and let b_i denote the homomorphic image of e^n_i. Then by (5.1) $O(b_i) \geqq O(a_i)$, Thus by (5.3.2) the mapping p: $a_i \to b_i$ can be extended to a homomorphism k. Since the homomorphism which induces Φ equals the product hk, it follows that $\Theta \leqq \Phi$. Therefore $\Theta = \Sigma O(a_i)$ and we arrive at the isomorphism statement of (5.3.3).

Finally, suppose that (5.3.3) holds and let $p_1, p_2 \in A^{(n)}$, $p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n)$ and let $b_1, \ldots, b_n$ be given as in 4.5.

Let h_1 and h_2 be the homomorphisms induced by the mappings $e^n_i \to a_i$ and $e^n_i \to b_i$ $(n = 1, \ldots, n)$, respectively, and Θ_1, Θ_2 the congruence relation of $\mathfrak{A}^{(n)}_{\mathcal{K}}$ induced by h_1 and h_2 respectively.

Then by (5.5.3) $\Theta_1 = \Sigma O(a_i)$ and (4.5.3) imply that Θ_2 satisfies (5.1) and, obviously, it satisfies (5.2) as well. Hence by the definition of $\Sigma O(a_i)$ we get $\Theta_1 \leqq \Theta_2$.

By the same argument as in 3.6 we get that there exists a homomorphism

$$h\colon ([a_1, \ldots, a_n]; F) \to ([b_1, \ldots, b_n]; F)$$

such that

$$a_i h = b_i \qquad (i = 1, \ldots, h).$$

Therefore

$$\begin{aligned} p_1(b_1, \ldots, b_n) &= p_1(a_1 h, \ldots, a_n h) = p_1(a_1, \ldots, a_n) h \\ &= p_2(a_1, \ldots, a_n) h = p_2(a_1 h, \ldots, a_n h) = p_2(b_1, \ldots, b_n), \end{aligned}$$

which was to be proved.

The proof of 5.3 is completed.

The only difficult notion involved in 5.3 is that of $\Sigma\theta_i$. It should be remarked that in case $\mathcal{K}$ has special properties $\Sigma\theta_i$ can be more simply characterized.

5.4. *Suppose $\mathcal{K}$ contains the homomorphic images and subalgebras of algebras in $\mathcal{K}$. Then $\Sigma\Theta_i$ always exists.*

6. Consequences and problems.

6.1. *A single element a is always weakly independent.*

Proof. Use the characterization given by (5.3.3) and 3.6.

6.2. *The elements $a_1, \ldots, a_n$ of an abelian group are independent if and only if $\Sigma k_i a_i = 0$ implies $k_1 a_1 = \ldots = k_n a_n = 0$.*

In this case $\mathfrak{A}^{(n)}_{\mathscr{K}}$ is the free abelian group on n generators, thus (5.5.3) gives the isomorphism:

$$([a_1, \ldots, a_n]; +) = \sum ([a_i]; +),$$

which is equivalent to the statement of 6.2. The same result is true for an arbitrary module over a ring.

6.3. *In lattices independence and weakly independence is the same.*

This is true by 4.6.

6.4. *If $a_1, \ldots, a_n$ is a weakly independent sequence, then $a_{i_1}, \ldots, a_{i_n}$ is also weakly independent, where $i_1, \ldots, i_n$ is any permutation of $1, \ldots, n$.*

Thus we can speak of a weakly independent set I, which in the finite case means that any ordering of I gives a weakly independent sequence while in the infinite case it means that every finite subset of I is weakly independent.

A *basis* H of an algebra $(A; F) \in \mathscr{K}$ is a set which is weakly independent and generates $(A; F)$.

6.5. *To every integer n there corresponds a class $\mathscr{K}$ and an algebra $(A; F) \in \mathscr{K}$ such that $(A; F)$ has a basis of k elements if and only if $k \leqslant n$.*

Let $p_1, p_2, \ldots, p_n$ be distinct primes, $\mathfrak{C}_i = (C_i; +)$ be the cyclic group of order p_i, $\mathfrak{C} = \Sigma \mathfrak{C}_i$, $\mathscr{K} = \{\mathfrak{C}\}$. Then $\mathscr{K}$ is effective in 6.5.

One of the unpleasant surprises about weak independence is that it is possible that $a_1, \ldots, a_n$ be independent and $a_1 \in [a_2, \ldots, a_n]$ as it is shown by the following example:

6.6. Let $A = \{a_1, a_2\}$, $F = \{f, g\}$ and $f(a_1) = a_1$, $f(a_2) = a_1$; $g(a_1) = a_2$, $g(a_2) = a_2$, and $\mathscr{K} = \{(A, F)\}$. Then $\mathfrak{A}^{(1)}_{\mathscr{K}}$ consists of three elements x, y, z and $f(x) = f(y) = f(z) = z$ and $g(x) = g(y) = g(z) = y$. It is easy to see that $x \equiv z(O(a_1))$, while $x \not\equiv z(O(a_1))$ and $x \equiv y(O(a_2))$ while $x \not\equiv y(O(a_2))$. Hence neither $O(a_1) \leqslant O(a_2)$ nor $O(a_1) \geqslant O(a_2)$ hold. Therefore, the only mapping p satisfying the assumption of (5.3.2) is p: $a_1 \to a_1$, $a_2 \to a_2$, whence a_1, a_2 is an independent sequence and $a_1 \in [a_2]$, $a_2 \in [a_1]$.

Some of the problems, which arise very naturally, are the following:

Problem 1. Let $n_1, n_2, \ldots$ be a (finite or infinite) sequence of integers. Construct a class $\mathscr{K}$ and an algebra $(A; F) \in \mathscr{K}$ such that $(A; F)$ has a basis of k elements if and only if $k = n_i$ for some i (**P 602**).

Problem 2. Let A be a set and J a hereditary family of finite subsets of A including all one element subsets. Prove the existence of an algebra $(A; F)$ (with $\mathscr{K} = \{(A; F)\}$) such that a subset in $(A; F)$ is weakly independent if and only if it is contained in J (**P 603**).

PROBLEM 3. Is it possible that an algebra without constant algebraic operations has a finite and also an infinite basis? Even if F is finite (**P 604**)?

PROBLEM 4. Find sufficient conditions on the class $\mathscr{K}$ under which $a_1 \in [a_2, \ldots, a_n]$ implies that $a_1, \ldots, a_n$ is not independent and not all elements are torsion free (**P 605**).

PROBLEM 5. Prove that if $\mathscr{K}$ is an equational class of algebras with anullary operation that determines a one-element subalgebra in every algebra in $\mathscr{K}$, and $a_1, \ldots, a_n$ is independent if and only if $[a_1, \ldots, a_n] = [a_1] \times \ldots \times [a_n]$, then $\mathscr{K}$ is equivalent to the class of all modules over a ring. (If this is not the case, what additional conditions are needed?) (**P 606**)

PROBLEM 6. Work out the notion which corresponds to the notion of p-rank in Abelian groups (**P 607**).

Remark. The role of elements of order p should be taken by elements whose order $O(a)$ is a dual atom in $\mathfrak{C}(A_{\mathscr{K}}^{(1)}; F)$.

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Reçu par la Rédaction le 29. 9. 1964