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REMARKS ON n-GROUPS AS ABSTRACT ALGEBRAS

BY

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1. In this paper we shall denote by $\mathfrak{A} = \langle A; \boldsymbol{F}\rangle$ an *algebra*, i.e. a non-empty set A and a class $\boldsymbol{F}$ of *fundamental operations* consisting of A-valued functions of several variables running over A. For $\boldsymbol{F} = \{f_1, f_2, \ldots, f_n\}$ we shall also write $\langle A; f_1, f_2, \ldots, f_n\rangle$. $\boldsymbol{A}(\mathfrak{A})$, or briefly $\boldsymbol{A}$, will denote a class of *algebraic operations* of the algebra $\mathfrak{A}$, that is the smallest class containing all fundamental operations and all *trivial operations* $e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k$ $(k = 1, 2, \ldots, n; n = 1, 2, \ldots)$, and closed with the respect to compositions. $\boldsymbol{A}^{(n)}(\mathfrak{A})$ $(n = 1, 2, \ldots)$, or briefly $\boldsymbol{A}^{(n)}$, will denote a subclass of $\boldsymbol{A}(\mathfrak{A})$ consisting of all algebraic operations of n variables. Further, by $\boldsymbol{A}^{(0)}$ we shall denote the set of values of constant algebraic operations. Elements belonging to $\boldsymbol{A}^{(0)}$ will be called *algebraic constants*. Two algebras $\mathfrak{A} = \langle A; \boldsymbol{F}_1\rangle$ and $\mathfrak{B} = \langle A; \boldsymbol{F}_2\rangle$ will be treated as identical if $\boldsymbol{A}(\mathfrak{A}) = \boldsymbol{A}(\mathfrak{B})$.

All the definitions formulated above are to be found in Marczewski's papers [4] and [5], where one can also find some related results.

Dörnte has introduced [1] the notion of an n-group, which is a natural generalization of the notion of a group. They were investigated by Post in [7], who calls them polyadic groups.

By an *n-group* we mean a set G with an n-ary operation defined in G which satisfies the following conditions:

1° for all $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n-1} \in G$,

$$\begin{aligned} f\big(f(x_1, x_2, \ldots, x_n), x_{n+1}, x_{n+2}, \ldots, x_{2n-1}\big) &= f\big(x_1, f(x_2, x_3, \ldots, x_n, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}\big) \\ &= \ldots = f\big(x_1, x_2, \ldots, x_{n-1}, f(x_n, x_{n+1}, \ldots, x_{2n-1})\big); \end{aligned}$$

2° for all $x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in G$ $(i = 1, 2, \ldots, n)$ there exists precisely one element $x_i \in G$ such that

$$f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = x_0.$$

We shall call condition 1°, which is a natural generalization of the associativiti in a group, the *associative law* in an n-group. Condition 2° is a generalization of the solvability of equations $ax = b$ and $ya = b$ in the group, and it defines n operations g_i $(i = 1, 2, .., n)$ inverse relative to f in G.

In particular, a 2-group is just a group. A special case of n-groups form also Prüfer's Schar [8].

It follows from the definition of an n-group that it may be conceived as an abstract algebra $\mathfrak{G} = \langle G; f, g_1, g_2, \ldots, g_n\rangle$ with $n+1$ fundamental operations.

If the operation f is symmetrical, i.e. invariant with respect to all permutations of arguments, then the n-group is called an *abelian n-group* (see [1]).

An element $\bar{x}$ which satisfies the equation

$$f(x, x, \ldots, x, \bar{x}) = x \tag{1}$$

is called *skew* to x.

One can prove that

$$f(x, x, \ldots, x, \bar{x}) = f(x, x, \ldots, \bar{x}, x) = \ldots = f(\bar{x}, x, \ldots, x, x), \tag{2}$$

and that

$$f(y, x, \ldots, \bar{x}, \ldots, x) = f(x, \ldots, \bar{x}, \ldots, x, y) = y \tag{3}$$

for all $x, y \in G$, where $\bar{x}$ can appear at any place under the sign of the function f.

It is possible to examine operations of m variables in n-groups, where $m > n$ and $m \equiv 1\big(\operatorname{mod}(n-1)\big)$, which are superpositions of the operation f. They have been called by Dörnte *long products*. By the associative law, two long products in which the same arguments appear in the same order are equal. The sign of the function f in a long m-ary product appears $(m-1)/(n-1)$ times.

By a *power* of an element x we shall mean

$$x^k = h_1(x, x, \ldots, x), \tag{i}$$

where h_1 is a long product of k arguments (obviously), $k \equiv 1\big(\operatorname{mod}(n-1)\big)$, or, respectively, x^{-k} is an element satisfying the condition

$$h_2(x^{-k}, x, x, \ldots, x) = x, \tag{ii}$$

where k is the positive integer and h_2 is a long product of $k+2$ elements $\big(k+2 \equiv 1\big(\operatorname{mod}(n-1)\big)\big)$.

It follows from the above that $\bar{x} = x^{2-n}$, $\overline{(\bar{x})} = x^{(n-2)^2}$, etc. Operations on powers are executed as for usual powers. The set of all powers of a given element x of an n-group forms the abelian n-group called a *cyclic n-group* generated by x [1].

2. The theorem below gives a possibility to define, for $n > 2$, an n-group with the aid of another set of axioms.

THEOREM 1. *A set G with an n-ary operation ($n > 2$) satisfying the associative law is an n-group if and only if for every $x \in G$ there exists an element $\bar{x} \in G$ such that for any $y \in G$ the following conditions are satisfied:*

$$f(\bar{x}, x, \ldots, x, y) = f(y, x, \ldots, x, \bar{x}) = y, \tag{4}$$

$$f(x, \bar{x}, \ldots, x, y) = f(y, x, \ldots, \bar{x}, x) = y. \tag{5}$$

Proof. It is well known that (3) follows from axioms of an n-group with an arbitrary localization of $\bar{x}$ under the sign of the function f, which implies, in particular, (4) and (5). Moreover, $\bar{x}$ is uniquely defined.

Now let us suppose that G is a set with an associative n-ary operation and suppose that for every $x \in G$ there exists an element $\bar{x} \in G$ such that for every $y \in G$ conditions (4) and (5) are satisfied. One needs only to prove that any one of the equations 2° is uniquely solvable in G. For that purpose let us examine the equation

$$f(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n) = x_0 \quad (i = 1, 2, \ldots, n). \tag{6}$$

Supposing that it is satisfied for a certain ξ, we have by (6)

$$\begin{aligned} & f(\underbrace{\bar{x}_{i-1}, x_{i-1}, \ldots, x_{i-1}}_{n-2}, \bar{x}_{i-2}, f(\underbrace{x_{i-2}, \ldots, x_{i-2}}_{n-3}, \bar{x}_{i-3}, x_{i-3}, \\ & f(\ldots, f(\ldots, f(\underbrace{x_1, \ldots, x_1}_{n-i}, f(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n), \underbrace{x_n, \ldots, x_n}_{i-2}, \\ & f(\underbrace{x_n, \ldots, x_n, \bar{x}_n}_{n-i}, \underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1}, f(\underbrace{x_{n-1}, \ldots, x_{n-1}, \bar{x}_{n-1}}_{n-i-1}, \ldots, f(\ldots, f(\ldots \\ & \qquad \ldots, f(x_{i+2}, \bar{x}_{i+2}, \underbrace{x_{i+1}, \ldots, \bar{x}_{i+1}}_{n-2})\underbrace{\ldots)\ldots)))\ldots)\ldots)))}_{n-2} \\ & = f(\underbrace{\bar{x}_{i-1}, x_{i-1}, \ldots, x_{i-1}}_{n-2}, \bar{x}_{i-2}, f(\underbrace{x_{i-2}, \ldots, x_{i-2}}_{n-3}, \bar{x}_{i-3}, x_{i-3}, f(\ldots, f(\ldots \\ & \ldots, f(\underbrace{x_1, \ldots, x_1}_{n-i}, x_0, \underbrace{x_n, \ldots, x_n}_{i-2}, f(\underbrace{x_n, \ldots, x_n, \bar{x}_n}_{n-i}, \underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1}, f(\ldots, f(\ldots \\ & \qquad \ldots, f(x_{i+2}, \bar{x}_{i+2}, \underbrace{x_{i+1}, \ldots, x_{i+1}, \bar{x}_{i+1}}_{n-2})\underbrace{\ldots)\ldots)))\ldots)\ldots)))}_{n-2}. \end{aligned} \tag{7}$$

Put

$$y = f(\underbrace{x_{n-1}, \ldots, x_{n-1}, \overline{x}_{n-1}}_{n-i-1}, \ldots, f(\ldots, f(\ldots$$
$$\ldots, f(x_{i+2}, \overline{x}_{i+2}, \underbrace{x_{i+1}, \ldots, x_{i+1}, \overline{x}_{i+1}}_{n-2} \underbrace{)\ldots)\ldots))}_{n-i-2}.$$

In view of the associative law and (4) we have:

$$f(\underbrace{x_1, \ldots, x_1}_{n-i}, f(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n), \underbrace{x_n, \ldots, x_n}_{i-2},$$
$$f(\underbrace{x_n, \ldots, x_n, \overline{x}_n}_{n-1}, \underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1}, y))$$
$$= f(\underbrace{x_1, \ldots, x_1}_{n-i-1}, f(x_1, f(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n),$$
$$\underbrace{x_n, \ldots, x_n, \overline{x}_n}_{n-2}), \underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1}, y)$$
$$= f(\underbrace{x_1, \ldots, x_1}_{n-i-1}, f(x_1, x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_{n-2},$$
$$f(x_{n-1}, \underbrace{x_n, \ldots, x_n, \overline{x}_n}_{n-1})), \underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1}, y)$$
$$= f(\underbrace{x_1, \ldots, x_1}_{n-i-1}, f(x_1, x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_{n-2}, x_{n-1}),$$
$$\underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1}, y).$$

Repeating a similar reasoning $n-i$ times we get

$$f(\underbrace{x_1, \ldots, x_1}_{n-i}, f(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n), \underbrace{x_n, \ldots, x_n}_{i-2},$$
$$f(\underbrace{x_n, \ldots, x_n, \overline{x}_n}_{n-i}, \underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1}, f(\ldots, f(\ldots, f(x_{i+2}, \overline{x}_{i+2},$$
$$\underbrace{x_{i+1}, \ldots, x_{i+1}, \overline{x}_{i+1}}_{n-2} \underbrace{)\ldots)\ldots)))}_{n-i} = f(\underbrace{x_1, \ldots, x_1}_{n-i+1}, x_2, \ldots, x_{i-1}, \xi).$$

Analogously, we also have

$$f(\underbrace{\overline{x}_{i-1}, x_{i-1}, \ldots, x_{i-1}}_{n-2}, \overline{x}_{i-2}, f(\underbrace{x_{i-2}, \ldots, x_{i-2}}_{n-3}, \overline{x}_{i-3}, x_{i-3}, f(\ldots, f(\ldots$$
$$\ldots, f(x_1, \ldots, x_1, x_2, \ldots, x_{i-1}, \xi)\ldots)\ldots))) = \xi.$$

And now it follows from (7) that

$$(8)\quad \xi = f(\underbrace{\bar{x}_{i-1}, x_{i-1}, \ldots, x_{i-1}}_{n-2}, \bar{x}_{i-2}, f(\underbrace{x_{i-2}, \ldots, x_{i-2}}_{n-3}, \bar{x}_{i-3}, x_{i-3},$$

$$f(\ldots, f(\ldots, f(\underbrace{x_1, \ldots, x_1}_{n-i}, x_0, \underbrace{x_n, \ldots, x_n}_{i-2}, f(\underbrace{x_n, \ldots, x_n, \bar{x}_n}_{n-i}, \underbrace{x_{n-1}, \ldots, x_{n-1}}_{i-1},$$

$$f(\ldots, f(\ldots, f(x_{i+2}, \bar{x}_{i+2}, \underbrace{x_{i+1}, \ldots, x_{i+1}, \bar{x}_{i+1}}_{n-2})\underbrace{\ldots)\ldots)))\ldots)\ldots)))}_{n-2}.$$

One can likewise verify the fact that the above obtained ξ satisfies (6) by making use of the associative law and of formula (5). If (6) possessed one more solution η, then repeating the above calculations we would obtain $\xi = \eta$.

It follows from the just proved theorem that an n-group ($n > 2$) can be also conceived as an abstract algebra with the following fundamental operations: one n-ary operation f and one unary operation (taking the skew element) satisfying 1° and equations (4) and (5), i. e. $\mathfrak{G} = \langle G; f, ^{-}\rangle = \langle G; f, g_1, g_2, \ldots, g_n\rangle$.

3. If $\mathfrak{G}$ is an n-group, then the set G taken with m-ary long product can be viewed as an m-group. We call any m-group isomorphic to that obtained in this way from an n-group, where $m > n$, and $m \equiv 1\big(\mathrm{mod}\ (n-1)\big)$ *reducible* to an n-group. The m-group will be said to be *irreducible* if it is not reducible to an n-group for any $n < m$. Dörnte [1] has proved that for every n there exists an irreducible n-group. One can observe that we have

THEOREM 2. *If an m-group $\mathfrak{G}_1 = \langle G; f, ^{-(f)}\rangle$ is reducible to an n-group $\mathfrak{G}_2 = \langle G; g, ^{-(g)}\rangle$, where $n > 2$, then $\boldsymbol{A}(\mathfrak{G}_1) \subset \boldsymbol{A}(\mathfrak{G}_2)$.*

Proof. Indeed, it suffices to show that the operation of taking the skew element in $\mathfrak{G}_1$ ($x \to \bar{x}^{(f)}$) is a superposition of operation g and that of taking the skew element in $\mathfrak{G}_2$ ($x \to \bar{x}^{(g)}$). Let g be an n-ary operation and f be an m-ary operation with $m = k(n-1)+1$. Then one can describe f as the superposition of operation g in the following way:

$$(9)\quad f(x_1, \ldots, x_m)$$
$$= g(g(\ldots g(g(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1})\ldots)x_{m-n+2}, \ldots, x_m).$$

Let

$$(10)\quad f_i(x_1, \ldots, x_{m-i(n-1)})$$
$$= g(g(\ldots g(x_1, \ldots, x_n)\ldots)x_{m-(i+1)(n-1)+1}, \ldots, x_{m-i(n-1)}),$$

where $i = 1, 2, \ldots, k-1$.

From equation (2), which takes the form of $x = f(\bar{x}^{(f)}, x, \ldots, x)$, we calculate $\bar{x}^{(f)}$. Suppose that $k < n-1$. Then by (9), (10), (1), and (3) we get

$$g(x, \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k+1}, \underbrace{x, \ldots, x}_{n-k-2})$$

$$= g(f(\bar{x}^{(f)}, x, \ldots, x), \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k+1}, \underbrace{x, \ldots, x}_{n-k-2})$$

$$= \underbrace{g(g(\ldots g}_{k+1}(\bar{x}^{(f)}, x, \ldots, x)\ldots)x, \ldots, x), \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k+1}, x, \ldots, x)$$

$$= g(g(f_1(\underbrace{\bar{x}^{(f)}, x, \ldots, x}_{m-n+1}), x, \ldots, x), \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k+1}, x, \ldots, x)$$

$$= g(f_1(\bar{x}^{(f)}, x, \ldots, x), g(x, \ldots, x, \bar{x}^{(g)}), \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k}, x, \ldots, x)$$

$$= g(f_1(\bar{x}^{(f)}, x, \ldots, x), x, \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k}, x, \ldots, x)$$

$$= g(g(f_2(\underbrace{\bar{x}^{(f)}, x, \ldots, x}_{m-2n+2}), x, \ldots, x), x, \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k}, x, \ldots, x)$$

$$= g(f_2(\bar{x}^{(f)}, x, \ldots, x), x, g(x, \ldots, x, \bar{x}^{(g)}), \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k-1}, x, \ldots, x)$$

$$= g(f_2(\bar{x}^{(f)}, x, \ldots, x), x, x, \underbrace{\bar{x}^{(g)}, \ldots, \bar{x}^{(g)}}_{k-1}, x, \ldots, x) = \ldots$$

$$= g(f_{k-2}(\underbrace{\bar{x}^{(f)}, x, \ldots, x}_{2n-1}), \underbrace{x, \ldots, x}_{k-2}, \bar{x}^{(g)}, \bar{x}^{(g)}, \bar{x}^{(g)}, x, \ldots, x)$$

$$= g(g(f_{k-1}(\bar{x}^{(f)}, x, \ldots, x), x, \ldots, x), \underbrace{x, \ldots, x}_{k-2}, \bar{x}^{(g)}, \bar{x}^{(g)}, \bar{x}^{(g)}, x, \ldots, x)$$

$$= g(f_{k-1}(\bar{x}^{(f)}, x, \ldots, x), \underbrace{x, \ldots, x}_{k-1}, \bar{x}^{(g)}, \bar{x}^{(g)}, x, \ldots, x)$$

$$= g(g(\bar{x}^{(f)}, x, \ldots, x), \underbrace{x, \ldots, x}_{k-1}, \bar{x}^{(g)}, \bar{x}^{(g)}, x, \ldots, x)$$

$$= g(\bar{x}^{(f)}, \underbrace{x, \ldots, x}_{k-1}, g(x, \ldots, x, \bar{x}^{(g)}), \bar{x}^{(g)}, x, \ldots, x)$$

$$= g(\bar{x}^{(f)}, \underbrace{x, \ldots, x}_{k}, \bar{x}^{(g)}, x, \ldots, x) = \bar{x}^{(f)}.$$

Therefore we have

$$\bar{x}^{(f)} = g(x, \bar{x},^{(g)} \ldots, \bar{x}^{(g)}, x, \ldots, x),$$

and so $\boldsymbol{A}(\mathfrak{G}_1) \subset \boldsymbol{A}(\mathfrak{G}_2)$.

Generally, let $[(k-1)/(n-2)]$ denote the integral part of $(k-1)/(n-2)$. Then $2 \leqslant k-(k-1)/(n-2)+1 < n-1$, and, as above, we obtain from (2) the following relations:

$$\underbrace{g(g(\ldots g(g}_{\left[\frac{k-1}{n-2}\right]+1}(x, \overline{x}^{(g)}, \ldots, \overline{x}^{(g)}), \overline{x}^{(g)}, \ldots, \overline{x}^{(g)}), \ldots), \underbrace{\overline{x}^{(g)}, \ldots, \overline{x}^{(g)}}_{k-\left[\frac{k-1}{n-2}\right]+1}, x, \ldots, x)$$

$$= \underbrace{g(g(\ldots g(g}_{\left[\frac{k-1}{n-2}\right]+1}(f(\overline{x}^{(f)}, x, \ldots, x), \overline{x}^{(g)}, \ldots, \overline{x}^{(g)}), \overline{x}^{(g)}, \ldots, \overline{x}^{(g)}), \ldots),$$

$$\underbrace{\overline{x}^{(g)}, \ldots, \overline{x}^{(g)}}_{k-\left[\frac{k-1}{n-2}\right]+1}, x, \ldots, x) = \underbrace{g(g(\ldots g(g(g(\ldots g(g}_{\left[\frac{k-1}{n-2}\right]+k+1}(\overline{x}^{(f)}, x, \ldots, x), \ldots), x, \ldots, x),$$

$$\overline{x}^{(g)}, \ldots, \overline{x}^{(g)}), \ldots), \underbrace{\overline{x}^{(g)}, \ldots, \overline{x}^{(g)}}_{k-\left[\frac{k-1}{n-2}\right]+1}, x, \ldots, x) = \overline{x}^{(f)}.$$

Thus $\boldsymbol{A}(\mathfrak{G}_1) \subset \boldsymbol{A}(\mathfrak{G}_2)$, q. e. d.

Obviously, if an m-group $\mathfrak{G}_1 = \langle G; f, ^- \rangle$ is reducible to a group $\mathfrak{G}_2 = \langle G; \cdot, ^{-1}, 1\rangle$, then $\overline{x} = x^{2-n}$, and so also $\boldsymbol{A}(\mathfrak{G}_1) \subset \boldsymbol{A}(\mathfrak{G}_2)$.

We note that $\overline{x}^{(g)} = x^{2-m} = x^{1+k(1-n)}$, which, by Theorem 2, shows that the powers with negative exponents defined above are algebraic operations in an n-group. Owing to this, to Theorem 1 and to the definition the powers in an n-group we obtain

COROLLARY 1. *The algebraic operations of one variable in an n-group are just powers x^q, where $q \equiv 1\big(\mathrm{mod}(n-1)\big)$.*

It is proved in [1] that an m-group is reducible to a group if and only if there exists an element r in it such that for any x we have

$$f(r, r, \ldots, r, x, r, \ldots, r) = x, \tag{11}$$

where x may appear at any possible place.

Dörnte has not given any conditions for an m-group to be reducible to an n-group. We shall prove below a kind of a criterion in a certain peculiar case.

THEOREM 3. *An m-group G with $m = n+k(n-1)^2$ $(k = 1, 2, \ldots)$ is reducible to an n-group, where $n > 2$, if and only if there exist in G elements r and s such that for arbitrary $x_1, x_2, \ldots, x_m \in G$ we have*

$$f(x_1, x_2, \ldots, x_m) = \underbrace{f(\ldots(f}_{(n-1)k+1}(x_1, \underbrace{\underbrace{r, \ldots, r, s}_{n-1}, \ldots, \underbrace{r, \ldots, r, s}_{n-1}}_{k}, x_2, \ldots \tag{12}$$

$$\ldots, x_{n-1}, \underbrace{\underbrace{r, \ldots, r, s}_{n-1}, \ldots, \underbrace{r, \ldots, r, s}_{n-1}}_{k}, x_n), \underbrace{\underbrace{r, \ldots, r, s}_{n-1}, \ldots, \underbrace{r, \ldots, r, s}_{n-1}}_{k}, x_{n+1}, \ldots$$

$$\ldots, x_{2n-2}, \underbrace{\underbrace{r, \ldots, r, s}_{n-1}, \ldots, \underbrace{r, \ldots, r, s}_{n-1}}_{k}, x_{2n-1}), \ldots),$$

$$\underbrace{\underbrace{r, \ldots, r, s}_{n-1}, \ldots, \underbrace{r, \ldots, r, s}_{n-1}}_{k}, x_{m-n+2}, \ldots, x_m),$$

where systems of elements $r, \ldots, r, s$ are inserted between elements x_i and x_{i+1} $(i = 1, 2, \ldots, m-1)$ k times.

Proof. First we note that the expression on the right-hand side of (12) is a long product in an m-group, because there are there $m+(n-1)k(m-1) \equiv 1(\mathrm{mod}(m-1))$ arguments and every function f has $n+k(n-1)^2$, i. e. precisely m arguments. Let $\mathfrak{G}_1 = \langle G; f, {}^{-(f)}\rangle$ be an m-group reducible to an n-group $\mathfrak{G}_2 = \langle G; g, {}^{-(g)}\rangle$, where g is a G-valued function of n variables running over G (it needs not to be an algebraic operation in $\mathfrak{G}_1$). Since $m = n+k(n-1)^2$, the operation f can be represented, by virtue of the associative, law as a superposition of operation g. Thus, for instance, the following long product is well defined:

$$(13)\qquad f(x_1, x_2, \ldots, x_m) = g(\underbrace{g(\ldots(g}_{k}(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}), \ldots)$$

$$x_{(k-1)(n-1)+2}, \ldots, x_{k(n-1)+1}), \ldots, \underbrace{g(\ldots(g}_{k}(x_{m-k(n-1)-2}, \ldots,$$

$$x_{m-(k-1)(n-1)-1}), \ldots) x_{m-n+1}, \ldots, x_{m-1}), x_m).$$

Therefore we have

$$(14)\qquad f(x_1, \underbrace{\underbrace{r, \ldots, r, s}_{n-1}, \underbrace{r, \ldots, r, s}_{n-1}, \ldots, \underbrace{r, \ldots, r, s}_{n-1}}_{k}, x_2, \ldots, x_{n-1},$$

$$\underbrace{\underbrace{r, \ldots, r, s}_{n-1}, \ldots, \underbrace{r, \ldots, r, s}_{n-1}}_{k}, x_n) = g(\underbrace{g(\ldots(g}_{k}(x_1, \underbrace{r, \ldots, r, s}_{n-1}), \underbrace{r, \ldots, r, s}_{n-1}), \ldots)$$

$$\underbrace{r, \ldots, r, s,}_{n-1})\ldots, \underbrace{g(\ldots(g}_{k}(x_{n-1}, \underbrace{r, \ldots, r, s}_{n-1}), \underbrace{r, \ldots, r, s}_{n-1}), \ldots)\underbrace{r, \ldots, r, s}_{n-1}), x_n).$$

Replacing now r by any element of $\langle G; g, {}^{-(g)}\rangle$ and s by the element $r^{-(g)}$ skew to r in $\mathfrak{G}_2$, we get, by virtue of (3) and (14),

$$(15)\qquad f(x_1, \underbrace{\underbrace{r, \ldots, r, r^{-(g)}}_{n-1}, \underbrace{r, \ldots, r, r^{-(g)}}_{n-1}, \ldots, \underbrace{r, \ldots, r, r^{-(g)}}_{n-1}}_{k}, x_2, \ldots, x_{n-1},$$

$$\underbrace{\underbrace{r, \ldots, r, r^{-(g)}}_{n-1}, \ldots, \underbrace{r, \ldots, r, r^{-(g)}}_{n-1}}_{k}, x_n) = g(x_1, x_2, \ldots, x_n).$$

Iterating this procedure on the right-hand side of equation (12) we check, in view of (13) and (15), its validity.

Conversely, let condition (12) be satisfied in an m-group $\mathfrak{G}_1$, where $m = n+k(n-1)^2$. We define in G a new n-ary operation (it needs not to be algebraic) by formula (15), substituting in it everywhere element s

instead of $\bar{r}^{(g)}$. As it is known from [1], G, with the n-ary operation defined by (15), is an n-group. Considering in it the m-ary long product appearing on the right-hand side of formula (13), we get, by virtue of (12) and the association, equation (13). The proof is complete.

Now we shall prove a

LEMMA. *An n-group $\mathfrak{G}_1 = \langle G; f, ^{-}\rangle$ reducible to a 2-group $\mathfrak{G}_2 = \langle G; \cdot, ^{-1}, 1\rangle$ is a group (i. e. $\boldsymbol{A}(\mathfrak{G}_1) = \boldsymbol{A}(\mathfrak{G}_2)$) if and only if it contains an algebraic constant.*

Proof. As it is known a necessary and sufficient condition for $\mathfrak{G}_1$ to be reducible to $\mathfrak{G}_2$ is the existence in G of an element r satisfying (11). In such a case we have $r = \bar{r}$, because $r = f(r, \ldots, r) = f(r, \ldots, \bar{r})$. Hence and from (4) and conditions (i) and (ii) defining the powers of the element we infer that $r^q = r$ for $q \equiv 1\big(\mathrm{mod}(n-1)\big)$. Let $\mathfrak{G}_1$ contain an algebraic constant c. Then there exists a unary algebraic operation g such that $g(x) = c$ for every x. Since, by virtue of Corollary 1, the only unary algebraic operations are the powers x^q, where $q \equiv 1\big(\mathrm{mod}(n-1)\big)$, we have $x^p = c$ for a certain p and every x. In particular, $r^p = c$. Hence, the element r being idempotent, it follows that $r = c$ (and c is the only algebraic constant in $\mathfrak{G}_1$). The operation

$$x \circ y = f(x, c, c, \ldots, c, y)$$

is an algebraic operation (as the superposition of c and f), and it is, as it is known from [1], a group-operation. The element c is a unity of this group, and the element y from the equation $f(x, c, c, \ldots, c, y) = c$ can be expressed, in view of Theorem 1, by elements x and c, as a superposition of operation f, and the taking the skew element. Obviously, set G with the operation $\circ$ is a group isomorphic to the group $\mathfrak{G}_2$. Forming now an n-ary long product we obtain the n-ary operation that we have started from. Thus fundamental operations of the algebra $\mathfrak{G}_2$ are contained in the set of algebraic operations of the algebra $\mathfrak{G}_1$. Since $\boldsymbol{A}(\mathfrak{G}_1) \subset \boldsymbol{A}(\mathfrak{G}_2)$, we have $\boldsymbol{A}(\mathfrak{G}_1) = \boldsymbol{A}(\mathfrak{G}_2)$.

We have thus proved that the condition is sufficient. The necessity is instantaneous, for the unity of the group $\mathfrak{G}_2$ is an algebraic constant in $\mathfrak{G}_1$.

THEOREM 4. *If an m-group $\mathfrak{G}_1 = \langle G; f, ^{-}\rangle$ has an algebraic constant, then it is a group $\mathfrak{G}_2 = \langle G; \cdot, ^{-1}, 1\rangle$.*

Proof. Let $\mathfrak{G}_1$ have algebraic constant c. Then there exists an integer q $\big(q \equiv 1\big(\mathrm{mod}(m-1)\big)\big)$ such that $x^q = c$ for every x. In particular, $c^q = c$ and $c = (\bar{c})^q = (c^{2-m})^q = (c^q)^{2-m} = c^{2-m} = \bar{c}$. Therefore $c = \bar{c}$, and, by virtue of (11), we infer that $\mathfrak{G}_1$ is an m-group reducible to the group in which the binary operation xy is defined by the formula $xy = f(x, c, \ldots, c, y)$. It follows then by the Lemma that $\mathfrak{G}_1$ is a group.

Now we give an example of a 3-group reducible to a group not having any algebraic constant.

Let $\langle K; \cdot, ^{-1}, 1\rangle$ denote the foury Klein's group. We consider K with the long product

$$x_1 \cdot x_2 \cdot x_3 = \sigma(x_1, x_2, x_3).$$

Since the Klein's group is abelian, σ is the symmetrical operation and $\mathfrak{G} = \langle K; \sigma, ^{-}\rangle$ is an abelian 3-group. It is quite easy to verify that

$$\sigma(x, y, z) \notin \{x, y, z\} \quad \text{for} \quad x \neq y \neq z \neq x$$

and

$$\sigma(x, y, y) = x; \quad \bar{x} = x.$$

The 3-group $\mathfrak{G}$ was considered by Świerczkowski in [9] (see also [5]). We have here $\boldsymbol{A}^{(0)}(\mathfrak{G}) = \emptyset$.

An algebra $\mathfrak{A}$ is called a *separable k variables algebra* ($k \geqslant 1$) if, for every pair $f, g \in \boldsymbol{A}^{(n)}$, where $n \geqslant k$, there exist operations $f_0 \in \boldsymbol{A}^{(k)}$ and $g_0 \in \boldsymbol{A}^{(n-k)}$ such that the equation

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$$

is equivalent in A to the equation

$$f_0(x_1, x_2, \ldots, x_k) = g_0(x_{k+1}, x_{k+2}, \ldots, x_n).$$

An algebra with separable k variables for every $k = 1, 2, \ldots$ will be called briefly *separable variables algebra*.

This notion has been introduced by Marczewski [3]. In [2] a theorem on the representation of these algebras has been proved. It turns out that they are the so-called quasi-linear algebras only (see [2]), and that the class of separable k variables algebras coincides with the class of algebras with separable k variables.

It is known that separable variables algebra contains always an algebraic constant. A group is a separable variables algebra if and only if it is abelian. Hence, by virtue of theorem 4, we get the following

COROLLARY 2. *An n-group is separable variables algebra if and only if it is an abelian group.*

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