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ON THE COMPLETION OF PARTIAL ALGEBRAS

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In the theory of partial algebras ([1]), the full or complete algebras play a rôle somewhat analogous to the rôle of compact spaces in the category of completely regular topological spaces ([2]). It might be considered as the subject of this paper to make this clear. We begin with

1. Partial Peano algebras. All algebras under consideration are of type $\Delta = (K_i)_{i\in I}$, i.e. we consider algebras (partial or full) (A, f), where $f = (f_i)_{i\in I}$ is a family of operations f_i: $D_i \subset A^{K_i} \to A$ (full if $D_i = A^{K_i}$, partial in general). (A, f) is a *partial Peano algebra of type* Δ *on set* M if the usual *Generalized Peano Axioms* hold true:

P1. $f_i(\mathfrak{a}) \notin M$, for all indices $i \in I$, for all sequences $\mathfrak{a} \in D_i$;

P2. $f_i(\mathfrak{a}) = f_j(\mathfrak{b})$ implies $i = j$ and $\mathfrak{a} = \mathfrak{b}$, for all indices $i, j \in I$, for all sequences $\mathfrak{a} \in D_i$, $\mathfrak{b} \in D_j$;

P3. $\overline{M} = A$,

where $\overline{M}$ (also $CM, C_A M, C_f M$, or most exactly $C_{(A,f)}M$) denotes the subalgebra (closed subset) generated by $M \subset A$. Mind that partial Peano algebras are allowed to be really partial, not full as the Peano algebras considered so far (the latter called *full* Peano algebras to avoid misunderstanding).

As a simple clue to the following, we state

THEOREM 1. *In any partial algebra* (A, f),

$$\overline{M} = M \cup \bigcup_{i\in I} f_i(\overline{M}^{K_i}),$$

for any subset $M \subset A$.

Proof. $M \cup \bigcup f_i(\overline{M}^{K_i}) \subset \overline{M}$ is clear, $\overline{M}$ containing M and being

([1]) For terminology and notation, cf. Słomiński [9], [10], [11] and Schmidt [8].

([2]) Some analogous results were obtained by A. Wojciechowska in the paper *Remarks on similarities and differences between the categories of topological spaces and quasi-algebras* written in the spring of 1965 at the suggestion of B. Węglorz on a seminar conducted by Dr. B. Gleichgewicht in Wrocław University. A part of Wojciechowska's paper can be found in a mimeographed report on that seminar *Seminarium z algebry* (in Polish), Wrocław 1966, p. 60-73. (*Note of the Editors.*)

closed with respect to fundamental operations. The converse inclusion follows from the obvious fact that $M \cup \bigcup f_i(\overline{M}^{Ki})$ is closed too.

Hence, the notion of partial Peano algebra may be given without reference to a set M, due to the

COROLLARY. *Let (A, f) be a partial Peano algebra on set M. Then*

$$M = A - \bigcup_{i \in I} \mathrm{Im} f_i.$$

One may call M the *Peano basis* of (A, f). This observation is helpful for

THEOREM 2. *Any relative algebra (B, g) of partial Peano algebra (A, f) is partial Peano* (³).

Proof. Since $g_i = f_i \cap (B^{K_i} \times B)$, P2 is trivial in (B, g). Let $N: = B - \bigcup \mathrm{Im}\, g_i$. We show that $(A - B) \cup \mathrm{C}_g N = A$, i.e. $x \in (A - B) \cup \mathrm{C}_g N$, by algebraic induction on $x \in A$. Let first be $x \in M: = A - \bigcup \mathrm{Im} f_i$. Then $x \notin \bigcup \mathrm{Im}\, g_i$, whence $x \in A - B$ or $x \in B - \bigcup \mathrm{Im}\, g_i = N$. Now, consider elements $a_\varkappa \in (A - B) \cup \mathrm{C}_g N$ $(\varkappa \in K_i)$ and $a = f_i(a_\varkappa | \varkappa \in K_i)$. In case $a \in A - B$, we are ready, therefore we may assume $a \in B$. If $a_\varkappa \in \mathrm{C}_g N$, for all $\varkappa \in K_i$, then $a \in \mathrm{C}_g N$, and we are ready again. If $a_\varkappa \notin \mathrm{C}_g N$, i.e. $a_\varkappa \in A - B$, for some $\varkappa \in K_i$, using P2 for algebra (A, f), we obtain $a \in N$. Hence $a \in (A - B) \cup \mathrm{C}_g N$ in any case, completing the proof.

THEOREM 3. *Let algebra A be partial, B full, φ an arbitrary mapping from abstract set A into abstract set B. Then φ is a homomorphism of algebras if and only if φ is a closed subset of product algebra $A \times B$* (⁴).

Proof. Let $f = (f_i)_{i \in I}$, $g = (g_i)_{i \in I}$ be the given algebraic structures on A and B respectively, let $h = (h_i)_{i \in I}$ be the product structure on $A \times B$. If φ is a homomorphism of (A, f) into (B, g), φ is a closed of subset $(A \times B, h)$. For consider elements $(a_\varkappa, b_\varkappa) \in \varphi$ $(\varkappa \in K_i)$, $(a, b) = h\big((a_\varkappa, b_\varkappa) | \varkappa \in K_i\big)$; then $b_\varkappa = \varphi(a_\varkappa)$ $(\varkappa \in K_i)$ and $a = f_i(a_\varkappa | \varkappa \in K_i)$, $b = g_i(b_\varkappa | \varkappa \in K_i)$, hence $\varphi(a) = g_i\big(\varphi(a_\varkappa) | \varkappa \in K_i\big) = g_i(b_\varkappa | \varkappa \in K_i) = b$ or $(a, b) \in \varphi$. It is in the opposite direction only that the assumption of completeness of algebra B is needed. Let φ be a closed subset of product algebra $A \times B$, assume $f_i(a_\varkappa | \varkappa \in K_i) = a$. By hypothesis, $b \in B$ exists such that $b = g_i(b_\varkappa | \varkappa \in K_i)$ where $b_\varkappa: = \varphi(a_\varkappa)$ $(\varkappa \in K_i)$. Then $(a, b) = h\big((a_\varkappa, b_\varkappa) | \varkappa \in K_i\big) \in \varphi$, hence $\varphi(a) = b$, showing that φ is a homomorphism.

This simple "closed-graph theorem" will be used as a nice clue for the proof of

THEOREM 4 (Recursion Theorem). *Let A be a partial Peano algebra. Then its Peano basis M is B-independent, for any full algebra B* (⁵).

(³) As a special case, any subalgebra of a (full) Peano algebra is (full) Peano; cf. Słomiński [9], chap. III, (1.5), Diener [1], Prop. 10.

(⁴) In the sense of Słomiński [11], § 2B: a full-homomorphism.

(⁵) For full Peano algebras, cf. Löwig [6], Theorem 2.16, Słomiński [9], chap. III, (1.3) (his proof, though incorrect, might be saved), [11], (2.8), Diener [1], Prop. 3; cf. also Karp [3], 3.3.4 and 8.2.2. The proof given here goes back to Lorenzen [5].

I.e. any mapping $\beta\colon M \to B$ may be extended to a (unique) homomorphism $\varphi\colon A \to B$.

Proof. Let φ be the subalgebra $\bar{\beta} \subset A \times B$ generated by $\beta \subset A \times B$. According to Theorem 1,

$$\varphi = \beta \cup \bigcup_{i \in I} h_i(\varphi^{K_i}), \tag{*}$$

where h_i are the product operations as in the last proof. We are going to show, by algebraic induction on $a \in A$: for any $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in \varphi$, i.e. φ is a mapping from A into B and therefore a homomorphic extension of β by Theorem 3. Inductive beginning: $a \in M$. Since $\beta \subset \varphi$, one has $\big(a, \beta(a)\big) \in \varphi$. Consider an arbitrary element $b \in B$ such that $(a, b) \in \varphi$. By (*), either $(a, b) \in \beta$, i.e. $b = \beta(a)$; or $(a, b) = h_i\big((a_\varkappa, b_\varkappa) \mid \varkappa \in K_i\big)$, hence $a = f_i(a_\varkappa \mid \varkappa \in K_i)$, contradicting P1. Inductive hypothesis: $a_\varkappa \in A$ $(\varkappa \in K_i)$, and there is exactly one $b_\varkappa \in B$ such that $(a_\varkappa, b_\varkappa) \in \varphi$ for all $\varkappa \in K_i$. By definition of φ, $\big(f_i(a_\varkappa \mid \varkappa \in K_i),\ g_i(b_\varkappa \mid \varkappa \in K_i)\big) \in \varphi$. Consider an arbitrary element $b \in B$ such that $\big(f_i(a_\varkappa \mid \varkappa \in K_i), b\big) \in \varphi$. By (*), either $\big(f_i(a_\varkappa \mid \varkappa \in K_i), b\big) \in \beta$, hence $f_i(a_\varkappa \mid \varkappa \in K_i) \in M$, contradicting P1; this leaves

$$\big(f_i(a_\varkappa \mid \varkappa \in K_i), b\big) = h_j\big((a_\varkappa^*, b_\varkappa^*) \mid \varkappa \in K_j\big) = \big(f_j(a_\varkappa^* \mid \varkappa \in K_j),\ g_j(b_\varkappa^* \mid \varkappa \in K_j)\big),$$

where $(a_\varkappa^*, b_\varkappa^*) \in \varphi$ $(\varkappa \in K_i)$; one obtains $f_i(a_\varkappa \mid \varkappa \in K_i) = f_j(a_\varkappa^* \mid \varkappa \in K_j)$, hence $i = j$, $a_\varkappa = a_\varkappa^* (\varkappa \in K_i)$ by P2, hence $b_\varkappa^* = b_\varkappa (\varkappa \in K_i)$ by inductive hypothesis, hence $b = g_i(b_\varkappa \mid \varkappa \in K_i)$, completing the proof.

Let us remark that the existence of partial Peano algebras is trivial: any *discrete algebra* A — all fundamental operations of which are empty by definition — is partial Peano; moreover, the discrete algebras are precisely the free algebras in the primitive class of all partial algebras. Various proofs have been given for the existence of full Peano algebras (of any type Δ, on any set M) [6]. This existence, together with Recursion Theorem 4 leads to the statement: the full Peano algebras are precisely the free algebras in the primitive class of all full algebras, often called the *absolutely free algebras*; accordingly, the full Peano algebra of type Δ on set M is unique up to a unique isomorphism over M (i.e. extending id_M) and will be denoted $P(\Delta, M)$. By Theorem 2, any relative algebra of $P(\Delta, M)$ is partial Peano; we shall prove the converse in the sequel.

2. The free completion of partial algebras. Given a partial algebra A, an *extension of* A in a very strict sense is any partial algebra B such that A is a relative algebra of B; then $\bar{A}$, the subalgebra of B generated by A

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[6] E.g. Słomiński [9], chap. III, (1.1), [11], (2.6); other proofs have been given by Löwig [7], Kerkhoff [4], and Harzheim [2].

is a *minimal extension of* A. A *completion of* A in a fairly wide sense is a full (complete) extension B of A; then $\bar{A}$ is a *minimal completion of* A. Adding a point $\infty \notin A$ to set A, there is an obvious "normal" method of making $A \cup \{\infty\}$ a completion of A; this *normal one-point completion of* A — by no means the only completion of A on set $A \cup \{\infty\}$! — is minimal if and only if A is really partial, not full, A being full if and only if A is its only minimal completion, in fact its only minimal extension. For it is by means of this unique (up to a unique isomorphism over A) normal one-point completion $\mathring{A}$, that one may easily conform the fact that partial algebra A is full if and only if A is a subalgebra of any of its extensions, or of its completions only.

THEOREM 5 (Existence of Free Completions). *Given any partial algebra A, there is a completion B of A, universal in the sense that, for any full algebra C and any homomorphism $\chi\colon A \to C$, there is exactly one homomorphism $\psi\colon B \to C$ extending χ. Completion B is unique up to unique isomorphism over A, and minimal.*

We call B the *universal* or *free completion of* A, denoted $\hat{A}$. It is an obvious analogue of the Stone-Čech compactification of a completely regular space.

Proof. By the well-known General Existence Theorem [7] there is a full algebra B_O and a homomorphism $\varphi_O\colon A \to B_O$, universal in the sense that each diagram of homomorphisms

$$\begin{tikzcd} A \arrow[rr, "\varphi_O"] \arrow[dr, "\chi"'] & & B_m \arrow[dl, dashed, "\psi_O"] \\ & C & \end{tikzcd}$$

(C full) can be filled in commutatively by a unique ψ_O. Taking $C = \mathring{A}$, the inclusion homomorphism $\chi = \mathrm{id}_A\colon A \to \mathring{A}$ is injective and strong [8]; therefore, as can easily be seen, φ_O is not only injective but strong too. General set theory enables us to find a set including set A as a subset,

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_O"'] & & B \arrow[dl, "\omega"] \\ & B_O & \end{tikzcd}$$

[7] Słomiński [10], Theorem 7, Schmidt [8], Theorem 2.

[8] Słomiński [11], § 1A. A surjective homomorphism is strong if and only if it induces the quotient structure (the weakest algebraic structure on its range such that it is a homomorphism); this is an obvious analogue of the strongly continuous mappings in the sense of Alexandroff-Hopf.

and a bijection ω: $B \to B_O$ extending φ_O (as a matter of fact, ω may be constructed in a canonical way, without using the Axiom of Choice!). B can be uniquely made a full algebra such that ω becomes an isomorphism. φ_O: $A \to B_O$ being a strong homomorphism, so is the inclusion id_A: $A \to B$: algebra A is a relative algebra of algebra B, hence B an extension, even completion of A in the strict sense of real inclusion. Equally, the universality property of φ_O: $A \to B_O$ is carried over to id_A: $A \to B$; the uniqueness of algebra B (up to a unique isomorphism over A) is an immediate consequence. From the proof of the General Existence Theorem or from the fact that the class of full algebras is closed with respect to subalgebras, one derives that extensions B_O and B are minimal.

Our main theorem gives an inner characterization of this free completion:

THEOREM 6 (Axiomatization of Free Completion). *Let* (B, g) *be a completion of partial algebra* (A, f). *Then* $B = \hat{A}$ *if and only the following Axioms of Free Completion hold true*:

FC1. $g_i(\mathfrak{b}) \in A$ *implies* $g_i(\mathfrak{b}) = f_i(\mathfrak{b})$, *for all indices* $i \in I$ *and for all sequences* $\mathfrak{b} \in B^{K_i}$;

FC2. $g_i(\mathfrak{b}) = g_j(\mathfrak{c}) \notin A$ *implies* $i = j$ *and* $\mathfrak{b} = \mathfrak{c}$, *for all indices* $i, j \in I$, *and for all sequences* $\mathfrak{b} \in B^{K_i}$, $\mathfrak{c} \in B^{K_j}$;

FC3. $\bar{A} = B$ (*completion* B *is minimal*).

Obviously, these axioms generalize the Peano Axioms P1-P3; as a matter of fact, one immediately obtains

COROLLARY 1. *B is a full Peano algebra if and only if B is the free completion of a discrete algebra.*

Thus, full Peano algebras constitute an analogue of Stone's space of ultrafilters on an abstract set.

Another immediate consequence:

COROLLARY 2. *A is a partial Peano algebra if and only if its free completion $\hat{A}$ is full Peano.*

COROLLARY 3. *The partial Peano algebras A are precisely the relative algebras of full Peano algebras B.*

Before proving Theorem 6, we have to construct — as is natural and inevitable for problems of this kind — a special completion of A in which FC1-FC3 are evident:

THEOREM 7. *Any partial algebra* (A, f) *has a completion* (B, g) *such that* FC1-FC3 *hold true.*

This is a generalization of the Existence Theorem for full Peano algebras and will be proved with the help of the latter:

Consider $(P, h) = P(\Delta, A)$, the full Peano algebra of type Δ on set A. We change the algebraic structure of P by the definition:

$$g_i^*(\mathfrak{b}) = \begin{cases} f_i(\mathfrak{b}) & \text{if } f_i(\mathfrak{b}) \text{ is defined,} \\ h_i(\mathfrak{b}) & \text{else.} \end{cases}$$

Algebra (P, g^*) is full too. Let (B, g) be the subalgebra of (P, g^*) generated by set A. By definition, $f_i(\mathfrak{a}) = a$ implies $g_i(\mathfrak{a}) = a$; moreover, if $g_i(\mathfrak{b}) = a \in A$ ($\mathfrak{b} \in B^{K_i}$), then, due to P1, $h_i(\mathfrak{b}) \neq a$, hence $g_i(\mathfrak{b}) = f_i(\mathfrak{b})$: (B, g) is an extension and therefore a completion of (A, f), and FC1 holds true, and so, by definition, does FC3. To prove FC2, assume $g_i(\mathfrak{b}) = g_j(\mathfrak{c}) \notin A$; then by definition $h_i(\mathfrak{b}) = g_i(\mathfrak{b}) = g_j(\mathfrak{c}) = h_j(\mathfrak{c})$, hence $i = j$ and $\mathfrak{b} = \mathfrak{c}$ by P2, completing the proof of Theorem 7.

We are now ready to prove Theorem 6. First, assume (B, g) to be a completion of (A, f) with properties FC1-FC3; we want to prove that (B, g) is the free completion of (A, g). This is an obvious generalization of the old Recursion Theorem for full Peano algebras, it will be proved with the help of our Recursion Theorem 4 for partial Peano algebras. We change the algebraic structure of B, restricting g_i to $B^{K_i} - D_i$, where D_i is the domain of f_i; then (B, h), where h_i is this restriction of g_i, is a partial algebra, in which P1 and P2 hold true, due to FC1 and FC2. Still, also P3 holds true, i.e. partial algebra (B, h) is generated by subset A, as was, by hypothesis FC3, the original algebra (B, g); in fact, $C_h A$ is a subalgebra of (B, g). Hence, (B, h) is a partial Peano algebra on set A, and our Recursion Theorem 4 may be applied. Let us consider a homomorphism χ from partial algebra (A, f) into an arbitrary full algebra (C, k). By Theorem 4, there is a homomorphism ψ from partial Peano algebra (B, h) into algebra (C, k) which extends χ; as χ is not an arbitrary map, but a homomorphism, one easily checks ψ to be a homomorphism even from full algebra (B, g) into (C, k): (B, g) is the free completion of (A, f).

Again, let (B, g) be an arbitrary free completion of (A, f). By Theorem 7, there is a completion (B', g') of (A, f) such that FC1-FC3 hold true. As we have just shown, (B', g') is a free completion too. Hence, both free completions are isomorphic over A, and FC1-FC3 also hold true in (B, g), completing the proof of Theorem 6.

Note that Theorem 6 and Theorem 7 give another proof of Theorem 5 that is independent of any more general existence theorem.

3. The semilattice of minimal completions. Given a certain partial algebra A, let B, C be any minimal completions of A. We write

$$B \succsim C \quad \text{or} \quad B \sim C$$

if there is a — necessarily unique — homomorphism (or isomorphism, respectively) over A, i.e. extending id_A, from B into — necessarily onto — C.

Then $\succeq$ is a quasi-ordering in the class of all minimal completions of A, $\sim$ its associated equivalence relation,

$$B \sim C \text{ if and only if } B \succeq C \text{ and } C \succeq B.$$

Moreover, from the universality property of $\hat{A}$,

$$\hat{A} \succeq B$$

for any minimal completion B; let φ_B be the unique homomorphism over A from $\hat{A}$ onto B, R_B its induced congruence relation on $\hat{A}$. By means of the homomorphism theorem,

$$B \succeq C \text{ if and only if } R_B \subset R_C,$$
$$B \sim C \text{ if and only if } R_B = R_C;$$

the classes of equivalent minimal completions of A correspond one-one with certain congruence relations on $\hat{A}$, and the quasi-ordering of minimal completions is just opposite to the inclusion of corresponding congruence relations.

THEOREM 8. *Let R be an arbitrary congruence relation on $\hat{A}$. $R = R_B$, for some minimal completion B, if and only if*

$$R \cap [A \times (A \cup \bigcup_{i \in I} g_i(A^{K_i})] = \mathrm{id}_A,$$

i.e. $\varphi(a) = \varphi(x)$, $a \in A$, $x \in A \cup \bigcup g_i(A^{K_i})$, implies $a = x$, where g_i are the fundamental operations of $\hat{A}$ and φ a homomorphism inducing R.

Proof. First, consider $R = R_B$, where (B, h) is a minimal completion of (A, f). Assume $a \in A$, $x \in A \cup \bigcup g_i(A^{K_i})$, $\varphi_B(a) = \varphi_B(x)$. If $x \in A$, $a = x$, since $\varphi_B(a) = a$, $\varphi_B(x) = x$. If $x = g_i(a_\varkappa \mid \varkappa \in K_i)$, $a_\varkappa \in A$ for all $\varkappa \in K_i$, we have

$$a = \varphi_B(a) = \varphi_B(x) = h_i\big(\varphi_B(a_\varkappa) \mid \varkappa \in K_i\big) = h_i(a_\varkappa \mid \varkappa \in K_i),$$

hence $a = f_i(a_\varkappa \mid \varkappa \in K_i)$, since (A, f) is a relative algebra of (B, h), hence $a = g_i(a_\varkappa \mid \varkappa \in K_i) = x$, since (A, f) is a relative algebra of $(\hat{A}, g)$.

Conversely, let congruence relation R fulfill the condition of our theorem. Then there is a full algebra B_O and a homomorphism φ_O from $\hat{A}$ onto B_O which induces R. By hypothesis, the restriction χ of φ_O to A is injective, so we may construct a set B including A and a bijection ω from B onto B_O which extends χ:

$$\begin{tikzcd}
A \arrow[r, "\subset"] \arrow[d, "\cap"'] \arrow[dr, "\chi"' near start] & \hat{A} \arrow[d, "\varphi_O"] \arrow[dl, "\varphi" near start] \\
B \arrow[r, "\omega"'] & B_O
\end{tikzcd}$$

B can be uniquely made a full algebra (B, h) such that ω becomes an isomorphism. Then $\varphi := \omega^{-1} \circ \varphi_O$ is a homomorphism from $\hat{A}$ onto B, which again induces the given congruence relation R; moreover, the restriction of φ onto A is nothing but id_A, hence id_A is a homomorphism from (A, f) into (B, h). (A, f) is even a relative algebra of (B, h). In fact, assume $a = h_i(a_\varkappa \mid \varkappa \in K_i)$, $a, a_\varkappa \in A\,(\varkappa \in K_i)$, and consider $x := g_i(a_\varkappa \mid \varkappa \in K_i) \in \hat{A}$. Then

$$\varphi(a) = a = h_i(a_\varkappa \mid \varkappa \in K_i) = h_i\big(\varphi(a_\varkappa) \mid \varkappa \in K_i\big) = \varphi(x);$$

by hypothesis, $a = x$ or $g_i(a_\varkappa \mid \varkappa \in K_i) = a \in A$. By FC1, we obtain $a = f_i(a_\varkappa \mid \varkappa \in K_i)$. So (B, h) is a completion of (A, f), moreover a minimal one, since set A generates algebra $\hat{A}$, and homomorphism φ is onto algebra B and carries set A onto A. Hence $\varphi = \varphi_B$ and $R = R_B$, completing the proof.

By Theorems 6 and 8, we have a comparatively concrete survey on the totality of minimal completions.

Note that the condition

$$R \cap (A \times A) = \mathrm{id}_A$$

will not be sufficient in Theorem 8. For let (A, h) be a full algebra, (A, f) a real partial algebra the algebraic structure f of which is weaker than the full structure h, i.e. id_A is a homomorphism from partial algebra (A, f) onto full algebra (A, h), not an isomorphism. There is a homomorphism φ of the free completion $\hat{A}$ of (A, f) onto (A, h) which extends id_A: for the induced congruence relation R, one has $R \cap (A \times A) = \mathrm{id}_A$. Still there is no minimal completion B of (A, f) such that $R = R_B$. For by the homomorphism theorem, there would be an isomorphism ω from (A, h) onto B such that $\varphi_B = \omega \circ \varphi$,

$$\begin{tikzcd}
(A,f) \arrow[r, "\subset"] \arrow[d, "\cap"'] \arrow[dr, "\subset"] & \hat{A} \arrow[dl, "\varphi"'] \arrow[d, "\varphi_B"] \\
(A,h) \arrow[r, "\omega"'] & B
\end{tikzcd}$$

But ω would have to be id_A, so algebra B would have to coincide with algebra (A, h) that is no extension of (A, f) at all.

An important application of Theorem 8: let us select a representative from each class of equivalent minimal completions. By Theorem 8, we obtain the

COROLLARY. *The representative minimal completions constitute a conditionally complete upper semilattice with greatest element* $\hat{A}$ (⁹).

I.e. for each non-empty family of (representative) minimal completions B_t there is a least upper bound. This is true because for the non-empty family of corresponding congruence relations R_t, there is the greatest lower bound, namely the intersection $\bigcap R_t$.

As we have stated above, this semilattice reduces to one element, A itself, if A is full. Let us therefore suppose A to be really partial. Then with certain trivial exceptions, our semilattice is not a lattice. In fact, let us consider its minimal elements. Clearly, any one-element completion of A is minimal. For let $B = A \cup \{b\}$, $b \notin A$ be a one-element completion (eo ipso a minimal extension), let C be an arbitrary minimal completion of A such that $B \succsim C$; then there is a homomorphism φ over A from B onto C which cannot but carry $\{b\} = B - A$ onto $C - A$, so φ is bijective, i.e. an isomorphism, hence $C \sim B$, and B is a minimal element in the semilattice of minimal completions. Still, if $A \neq \emptyset$ and type Δ is such that $K_j \neq \emptyset$ for some index $j \in I$, then there are at least two non-isomorphic one-element completions of A. In fact, take the normal one-element completion $\mathring{A} = A \cup \{\infty\}$ and change its algebraic structure g, at least the operation g_j, by selecting an element $a^* \in A$ and defining

$$h_j(a_\varkappa \mid \varkappa \in K_j) = \begin{cases} a^* & \text{if } a_\varkappa = \infty \text{ for some } \varkappa \in K_j, \\ g_j(a_\varkappa \mid \varkappa \in K_j) & \text{elsewhere}. \end{cases}$$

Then (A, h) is a one-element completion such that $(A, h) \nsim (A, g)$. Unfortunately, the situation is of still greater complexity. For instance, consider algebra (B, g), where B is a set of four elements a_1, a_2, b_1, b_2, and $g\colon B \to B$ is the full operation of one variable defined by $g(a_\nu) = b_\nu$, $g(b_\nu) = a_\nu$ $(\nu = 1, 2)$. Consider the discrete relative algebra (A, f), where $A = \{a_1, a_2\}$. Then (B, g) is a minimal completion of (A, f) which is a minimal element in our semilattice of minimal completions. For let (C, h) be a minimal completion of (A, f) such that $(B, g) \succsim (C, h)$; since $h(\varphi(b_\nu)) = \varphi(g(b_\nu)) = \varphi(a_\nu) = a_\nu (\nu = 1, 2)$, we have $\varphi(b_1) \neq \varphi(b_2)$, moreover, as (A, f) is a discrete relative algebra of (C, h), $\varphi(b_\nu) \notin A (\nu = 1, 2)$, showing that φ is injective, i.e. an isomorphism: $(B, g) \sim (C, h)$. Let us finally remark that in the case of a finitary type Δ (all index sets K_i finite, i.e. all fundamental operations finitary), we may conclude that our semilattice is weakly atomistic in the sense that each minimal com-

(⁹) By a conditionally complete ordered set, we want to understand a (partially) ordered set P such that each non-empty bounded from above family has the least upper bound; equivalently: each non-empty bounded from below family has the greatest lower bound; equivalently: P can be made a complete lattice by adjunction of zero and only one. This property is stronger than the notion given by G. Birkhoff.

pletion B contains a minimal completion C, $B \supsetneq C$, that is a minimal element of our semilattice of minimal completions. For Theorem 8 shows that the union of a chain of congruence relations R_B is of the same kind; by Kuratowski-Zorn's lemma, each R_B is contained in a maximal congruence relation R_C.

4. The complete lattice of normal minimal completions. The situation becomes nice if we restrict ourselves to those minimal completions that behave well. In general, an extension (B, g) of partial algebra (A, f) may be called *normal* (or perhaps "correct", if one dislikes using the abused word "normal" for the $(n+1)$-st time) if FC1 holds true; as (A, f) is a relative algebra of (B, g), it will be sufficient to postulate that

$$g_i(\mathfrak{b}) \in A \ (\mathfrak{b} \in B^{K_i}) \text{ implies } \mathfrak{b} \in A^{K_i},$$

for all indices $i \in I$.

THEOREM 9. *Minimal completion B is normal if and only if*

$$R_B \cap (A \times \hat{A}) = \mathrm{id}_A.$$

Since id_A is the restriction of φ_B to A, this condition (strengthening the condition of Theorem 8) means that $\varphi_B(x) \in A$ implies (and is implied by) $\varphi_B(x) = x$, or simply $x \in A$, for all $x \in \hat{A}$. This is helpful for the

Proof of Theorem 9. Let (B, h) be normal. We prove the latter implication by algebraic induction on x. The inductive beginning $x \in A$ is trivial. Inductive hypothesis: $\varphi_B(x_\varkappa) \in A$ implies $x_\varkappa \in A$, for all $\varkappa \in K_i$, for some index $i \in I$, some family $(x_\varkappa)_{\varkappa \in K_i}$ of elements $x_\varkappa \in \hat{A}$. We have to prove our implication for $x := g_i(x_\varkappa \mid \varkappa \in K_i)$ (g the algebraic structure of $\hat{A}$). So assume $\varphi_B(x) \in A$, i.e. $h_i(\varphi_B(x_\varkappa) \mid \varkappa \in K_i) \in A$. Since (B, h) is normal, $\varphi_B(x_\varkappa) \in A$, even $h_i\big(\varphi_B(x_\varkappa) \mid \varkappa \in K_i\big) = f_i\big(\varphi_B(x_\varkappa) \mid \varkappa \in K_i\big)$ (f the algebraic structure of A), by inductive hypothesis, $x_\varkappa \in A$, i.e. $\varphi_B(x_\varkappa) = x_\varkappa$, for all $\varkappa \in K_i$. So $f_i(x_\varkappa \mid \varkappa \in K_i)$ is defined, and since (A, f) is a relative algebra of $(\hat{A}, g)$, $x = g_i(x_\varkappa \mid \varkappa \in K_i) = f_i(x_\varkappa \mid \varkappa \in K_i) \in A$.

Conversely, let minimal completion (B, h) fulfill our condition. Assume $h_i(b_\varkappa \mid \varkappa \in K_i) \in A$. By the Axiom of Choice, there are elements $x_\varkappa \in \hat{A}$ such that $b_\varkappa = \varphi_B(x_\varkappa)$, for all $\varkappa \in K_i$, and we have $\varphi_B\big(g_i(x_\varkappa \mid \varkappa \in K_i)\big) = h_i(b_\varkappa \mid \varkappa \in K_i) \in A$. By assumption, $g_i(x_\varkappa \mid \varkappa \in K_i) \in A$; by FC1 (for $\hat{A}$), $b_\varkappa = \varphi_B(x_\varkappa) = x_\varkappa \in A$, for all $\varkappa \in K_i$, completing the proof.

Since relation $R_B \subset \hat{A} \times \hat{A}$ is reflexive and symmetric, the condition of Theorem 9 is equivalent with

$$R_B \subset \mathrm{id}_A \cup \big((\hat{A} - A) \times (\hat{A} - A)\big);$$

or, as

$$\mathrm{id}_A \cup \big((\hat{A} - A) \times (\hat{A} - A)\big) = R_{\mathring{A}},$$

our condition means that

$$R_B \subset R_{\mathring{A}}.$$

This again makes evident (which was trivial by definition of $\mathring{A}$) that the normal one-point completion $\mathring{A}$ is a normal extension in the general sense given here. Moreover, we have the

COROLLARY 1. *Minimal completion B is normal if and only if*

$$B \succeq \mathring{A}.$$

And, which is more important, we have the following nice counterpart to the Corollary of Theorem 7:

COROLLARY 2. *The representative normal minimal completions constitute a complete lattice with the greatest element $\hat{A}$ and the least element $\mathring{A}$.*

Note that all minimal completions are normal in the trivial case that type Δ consists of constants only and $K_i = \emptyset$ for all indices $i \in I$.

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