COLLOQUIUM MATHEMATICUM

VOL. XVII 1967 FASC. 2

ON A NEW NOTION OF INDEPENDENCE IN UNIVERSAL ALGEBRAS

BY

G. GRÄTZER (UNIVERSITY PARK, PENNSYLVANIA, U.S.A.)

1. Introduction. Two different notions of independence are used in abelian group theory. The classical notion is the following: the elements $a_1, \ldots, a_k$ of an abelian group are independent if

$$\sum_{i=1}^{k} n_i a_i = 0 \text{ implies } n_1 = n_2 = \ldots = n_k = 0; \tag{1.1}$$

hence a single element a is independent if and only if it is torsion free.

In recent papers a new notion of independence (introduced by T. Szele) has frequently been used:

the elements $a_1, \ldots, a_k$ are independent if

$$\sum_{i=1}^{k} n_i a_i = 0 \text{ implies } n_1 a_1 = \ldots = n_k a_k = 0. \tag{1.2}$$

Hence a single element a is always independent (see e.g. [2]).

The first notion is connected with the notion of free abelian groups.

The notion of a free universal algebra was introduced by Birkhoff [1] and based on this Marczewski [4] gave a general notion of independence in universal algebras.

In this note an attempt will be made to generalize Marczewski's notion of independence in such a way that when applied to abelian groups it should be indentical with (1.2).

This will be achieved by defining the order of an element in a universal algebra.

The basic notions are given in § 2, the order of an element is defined in § 3 while in § 4 the new notion of independence is given. The characterization theorem of weak independence is proved in § 5. Some of its consequences and several unsolved problems are listed in § 6.

It should be noted that all the notions introduced in § 2 are standard ones and are given here only for completeness sake. However, the notion of the order of an element — however evident it is — seems to be new.

Most of the results of this paper were contained in my mimeographed note [3], which had a limited distribution in 1962.

2. Some notions and notation. An *algebra* is a couple $(A; F)$ where A is a set and F is a collection of fundamental operations. Every operation $f \in F$ is finitary, $f = f(x_1, \ldots, x_n)$ (n is an integer and depends on f), which means that if $(a_1, \ldots, a_n)$ is an n-tuple of elements of A, then $f(a_1, \ldots, a_n)$ is a well defined element of A.

Let $B \subseteq A$; we call $(B; F)$ a *subalgebra* of $(A; F)$ if $a_1, \ldots, a_n \in B$ and $f = f(x_1, \ldots, x_n) \in F$ imply $f(a_1, \ldots, a_n) \in B$.

Let $(A; F)$ and $(B; F)$ be algebras and h: $x \to xh$ a many-one mapping of A into B. The mapping h is called a *homomorphism* if

$$f(x_1, \ldots, x_n)h = f(x_1 h, \ldots, x_n h)$$

holds identically for every $f \in F$. Accordingly, an *isomorphism* h is a homomorphism which is one-to-one and onto ($Ah = B$); an *endomorphism* is a homomorphism of $(A; F)$ into itself, an *automorphism* is an isomorphism of $(A; F)$ with itself.

A congruence relation Θ on $(A; F)$ is an equivalence relation on A which has the substitution property:

(SP) if $a_i \equiv b_i(\Theta)$, $i = 1, 2, \ldots, n$, then $f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)(\Theta)$ for every $f \in F$.

Let A/Θ denote the set of equivalence classes modulo Θ and a/Θ ($a \in A$) the equivalence class represented by a. Then $(A/\Theta; F)$ is an algebra where for every $f \in F$ we put

$$f(a_1/\Theta, \ldots, a_n/\Theta) = f(a_1, \ldots, a_n)/\Theta.$$

The set of all congruence relations on $(A; F)$ is denoted by $C(A; F)$.

Let $\Theta_1, \Theta_2 \in C(A; F)$. We put $\Theta_1 \leqslant \Theta_2$ if $x \equiv y(\Theta_1)$ implies $x \equiv y(\Theta_2)$. This makes $C(A; F)$ a partially ordered set; it can be easily proved that the l.u.b.: $\Theta_1 \cup \Theta_2$ and g.l.b.: $\Theta_1 \cap \Theta_2$ always exist. $\mathfrak{C}(A; F) = \big(C(A; F); \cup, \cap\big)$ is a lattice, it is called the *congruence lattice* of $(A; F)$.

The class $A^{(n)} (n = 1, 2, \ldots)$ of algebraic operation of n-variables is the smallest class satisfying the following two conditions:

(2.1) the trivial operations e_i^n defined by $e_i^n(x_1, \ldots, x_n) = x_i (i = 1, 2, \ldots n)$ are in $A^{(n)}$;

(2.2) if $g_1, \ldots, g_k \in A^{(n)}$ and $f = f(x_1, \ldots, x_k) \in F$, then $f(g_1, \ldots, g_k) = f\big(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\big)$ is also in $A^{(n)}$.

Let $\mathscr{K}$ be a fixed class of algebras $(A; F)$.

An equivalence relation on $A^{(n)}$ is defined as follows: let $f, g \in A^{(n)}$; we write $f \equiv g$ if $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$ for every $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$.

Let $A_{\mathscr{K}}^{(n)}$ describe the equivalence classes under this equivalence relation. We can define the operations on $A_{\mathscr{K}}^{(n)}$ in a natural way; formula (2.2) shows that $(A_{\mathscr{K}}^{(n)}; F)$ is an algebra; this will be denoted by $\mathfrak{A}_{\mathscr{K}}^{(n)}$.

We put $A^{(\omega)} = A^{(0)} \cup A^{(1)} \cup A^{(2)} \cup \ldots$ and we define an equivalence: by $f \equiv g$ $(f \in A^{(k)}, g \in A^{(l)})$ it for every $a_1, a_2, \ldots, a_{\max(k,l)} \in A$, $(A; F) \in \mathscr{K}$ the equality $f(a_1, \ldots, a_k) = g(a_1, \ldots, a_l)$ holds. The equivalence classes will be denoted by $A_{\mathscr{K}}^{(\omega)}$ and the corresponding algebra $(A_{\mathscr{K}}^{(\omega)}; F)$ by $\mathfrak{A}_{\mathscr{K}}^{(\omega)}$.

Let $H \subseteq A$; we define the subset $[H]$ of A by $a \in [H]$ if there exists an integer n, and $f \in A^{(n)}$ and $h_1, \ldots, h_n \in H$ such that $f(h_1, \ldots, h_n) = a$.

Then $([H]; F)$ is a subalgebra of $(A; F)$; it is the subalgebra generated by H.

If A, B are sets, $A - B$ denotes the set theoretical difference. $\{a_1, \ldots, a_n\}$ denotes the set whose elements are $a_1, \ldots, a_n$. The notation $[\{a_1, \ldots\}]$ is replaced by $[a_1, \ldots]$.

3. The order of an element. Let $\mathscr{K}$ be a class of algebras, $(A; F) \in \mathscr{K}$, $a \in A$. The order of a is defined as follows:

Consider the mapping

$$e_1^1 \to a;$$

this has a unique extension to a homomorphism h of $\mathfrak{A}_{\mathscr{K}}^{(1)}$ into $(A; F)$; let $O(a)$ denote the congruence relation induced by h; we call $O(a)$ the *order* of a.

It is obvious that $O(a)$ is uniquely determined by a, $(A; F)$ and $\mathscr{K}$. Further, $O(a) \in C(A_{\mathscr{K}}^{(1)}; F) = C(\mathfrak{A}_{\mathscr{K}}^{(1)})$.

We first give a few examples:

3.1. Let $\mathscr{K}$ be the class of all additive groups. Then $\mathfrak{A}_{\mathscr{K}}^{(1)}$ is isomorphic to the group $\mathfrak{J}$ of integers, let $e_1^1 \to 1$ under this isomorphism. Let $\mathfrak{G} \in \mathscr{K}$, $a \in G$. Then the mapping $1 \to a$ has a unique extension to a homomorphism of $\mathfrak{J}$ into $\mathfrak{G}$. It is easy to see, $O(a)$ is the congruence modulo n, where n is the least integer with $na = 1$. This $O(a)$ is completely described if we give this n, which is usually called the order of a.

3.2. Let $\mathscr{K}$ be the class of all semi-groups. Now $\mathfrak{A}_{\mathscr{K}}^{(1)}$ is isomorphic to $\mathfrak{N}$, the (additive) semi-group of positive integers, again $e_1^1 \to 1$ under this isomorphism. In this case $O(a)$ can be described by a pair of non-negative integers (m, n) as follows: $x \equiv y$ $\big(O(a)\big)$ $(x, y \in I)$ if and only if $x = y$ or $x > m$, $y > m$ and n divides $x - y$.

3.3. $\mathscr{K}$ is the class of all right modules over a ring $(R; +, \cdot)$. This may also be included in the above discussion in the usual way by making every element $r \in R$ correspond to a unary operation f_r and put F

$=\{+, f_r\}_{r\in R}$ and considering a right-module M as an algebra $(M; F)$. Then $\mathfrak{A}^{(1)}_{\mathscr{K}}$ *is isomorphic to* $(R; F)$ *and* $O(a)$ *may be identified with the* class containing the zero of $\mathfrak{R}$, which is an ideal I_a. Usually, this ideal I_a is called the order of a.

These examples show that the notion of an order of an element is a natural generalization of known concepts.

The following propositions show the usefulness of this notion:

3.4. *Let* $(A; F)$, $(B; F)\in\mathscr{K}$ *and* h *be a homomorphism of* $(A; F)$ *into* $(B; F)$, $a\in A$.

Then

$$O(a) \leqq O(ah). \tag{3.5}$$

To prove this we consider the homomorphisms:

$$h_1\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (A; F);\ e_1^1 \to a;$$

$$h_2\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (B; F);\ e_1^1 \to ah.$$

Then

$$h_1 h = h_2,$$

which implies that $x \equiv y\big(O(ah)\big)$ if and only if $xh_2 = yh_2$, i.e. if $(xh_1)h = (yh_1)h$. Thus $xh_1 = yh_1$, implies $xh_2 = yh_2$, i.e. $x \equiv y\big(O(a)\big)$ implies $x \equiv y\big(O(ah)\big)$.

A partial converse of 3.4 holds too:

3.6. *Let* $(A; F)$, $(B; F)\in\mathscr{K}$, $a\in A$, $b\in B$ *and suppose* $O(a)\leqq O(b)$. *Then there exists a homomorphism*

$$h\colon ([a]; F) \to ([b]; F),$$

carrying a *into* b $(b = ah)$.

To prove this consider the homomorphism:

$$h_1\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (A; F),\ e_1^1 \to a;$$

$$h_2\colon \mathfrak{A}^{(1)}_{\mathscr{K}} \to (B; F),\ e_1^1 \to b.$$

We define h as follows: let $a_1\in[a]$, then there exists an $a_2\in A^{(1)}_{\mathscr{K}}$ with $a_1 = a_2 h_1$; let $a_1 h = a_2 h_2$.

First we have to prove that h is uniquely defined. Indeed, if $a_2 h_1 = a_3 h_1$ $(a_3\in A^{(1)}_{\mathscr{K}})$, then $a_2 \equiv a_3\big(O(a)\big)$, which implies $a_2 \equiv a_3\big(O(b)\big)$, i.e. $a_2 h_2 = a_3 h_2$.

To show the substitution property let $f\in F$, $f = f(x_1, \ldots, x_n)$, take $a_1, \ldots, a_n\in[a]$; then

$$a_i = p_i(a), \quad p_i\in A^{(1)}, \quad i = 1, \ldots, n.$$

We want to show

$$f(a_1, \ldots, a_n)h = f(a_1h, \ldots, a_nh).$$

Let

$$a_i' \in A_{\mathscr{K}}^{(1)}, \quad a_i'h_1 = a_i, \quad i = 1, \ldots, n,$$

$$c \in A_{\mathscr{K}}^{(1)}, \quad ch_1 = f(a_1, \ldots, a_n).$$

Then

$$c \equiv f(a_1', \ldots, a_n')\big(O(a)\big),$$

thus

$$c \equiv f(a_1', \ldots, a_n')\big(O(b)\big),$$

and also

$$ch_2 = f(a_1', \ldots, a_n')h_2.$$

Thus

$$f(a_1, \ldots, a_n)h = ch_2 = f(a_1', \ldots, a_n')h_2 = f(a_1'h_2, \ldots, a_n'h_2)$$
$$= f(a_1h, \ldots, a_nh).$$

It should be noted that 3.6 is new only in this form. In fact it is a special case of the so called Second Isomorphism Theorem, which is a part of the folklore.

3.7. The order of an element (a, b), in the direct product of $(A; F)$ and $(B; F)$, can be computed as follows:

$$O(a, b) = O(a) \frown O(b), \tag{3.8}$$

if (A, F), (B, F) and $(A \times B, F)$ are in $\mathscr{K}$.

Let $p_1, p_2 \in A_{\mathscr{K}}^{(1)}$. (3.8) means that $p_1 \equiv p_2\big(O(a, b)\big)$ if and only if $p_1 \equiv p_2\big(O(a)\big)$ and $p_1 \equiv p_2\big(O(b)\big)$. Since $p_1 \equiv p_2\big(O(a, b)\big)$ means $p_1\big((a, b)\big) = p_2\big((a, b)\big)$ and so on, we get that we have to prove the following: $p_1\big((a, b)\big) = p_2\big((a, b)\big)$ if and only if $p_1(a) = p_2(a)$ and $p_1(b) = p_2(b)$, which holds by definition.

4. Independence and weak independence. Marczewski's notion of independence is defined as follows:

4.1. Let $\mathscr{K}$ be a class of algebras, $(A; F) \in \mathscr{K}$, $a_1, \ldots, a_n \in A$. We say that the sequence $a_1, \ldots, a_n$ is *independent* if

$$p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n), \quad p_1, p_2 \in A^{(n)},$$

imply

$$p_1 \equiv p_2.$$

It may be remarked that Marczewski's definition is restricted to the case when $\mathscr{K}$ consists only of $(A; F)$; some of his results, however, remain true for an arbitrary class $\mathscr{K}$. The characterization theorem of independent sequences is the following:

4.2. *Let $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$. Then the following conditions are equivalent:*

(4.2.1) *$a_1, \ldots, a_n$ is an independent sequence;*

(4.2.2) *let $b_1, \ldots, b_n \in B$, $(B; F) \in \mathscr{K}$ and $p\colon a_i \to b_i$, $i = 1, \ldots, n$. Then p can be extended to a homomorphism of $([a_1, \ldots, a_n]; F)$ into $(B; F)$;*

(4.2.3) *the mapping $p\colon e_i^n \to a_i$ can be extended to an isomorphism h of $\mathfrak{A}_{\mathscr{K}}^{(n)}$ onto $([a_1, \ldots, a_n]; F)$.*

The equivalence of (4.2.1) and (4.4.2) is stated in [4]; I am sure that Marczewski knows that they are equivalent to (4.2.3) as well, however, I cannot give a reference.

An important corollary of 4.2 (which is also due to Marczewski) is:

4.3. *If $a_1, \ldots, a_n$ is independent, then so is $a_{i_1}, \ldots, a_{i_n}$, where $j \to i_j$ is any permutation of $1, \ldots, n$.*

Thus we can speak of an independent set $a_1, \ldots, a_n$, because the ordering does not matter.

4.4. *An element a is independent if and only if a is torsion free, i.e. $O(a) = \omega$.*

This is trivial by (4.2.3) and the definition of $O(a)$.

Now we give the definition of weak independence.

4.5. Let $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$. We say that the sequence $a_1, \ldots, a_n$ is *weakly independent* if

$$p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n), \quad p_1, p_2 \in A^{(n)}, \tag{4.5.1}$$

imply

$$p_1(b_1, \ldots, b_n) = p_2(b_1, \ldots, b_n) \tag{4.5.2}$$

for every $b_1, \ldots, b_n \in B$, $(B; F) \in \mathscr{K}$, for which

$$O(a_i) \leqslant O(b_i), \quad i = 1, \ldots, n. \tag{4.5.3}$$

First, let us see some trivial consequences of this definition.

4.6. *Suppose $a_1, \ldots, a_n$ are torsion free elements. Then $a_1, \ldots, a_n$ is independent if and only if it is weakly independent.*

The difference between independence and weak independence is condition (4.5.3). However, if $O(a_1) = \ldots = O(a_n) = \omega$, then (4.5.3) is no restriction on the choice of the b_i and hence in this case the two notions are equivalent.

4.7. *If $\mathscr{K}$ is a subclass of lattices, independence and weak independence are equivalent.*

Obviously, since in a lattice every element is torsion free.

5. Characterizations of weak independence. We would like to get a result analogous to 4.2. In order to achieve that we need some notation.

The algebra $\mathfrak{A}_{\mathscr{K}}^{(n)}$ is generated by $e_1^n, \ldots, e_n^n$, and the subalgebra $\mathfrak{A}_i$ generated by e_i^n is isomorphic to $\mathfrak{A}_{\mathscr{K}}^{(1)}$. Suppose we are given n congruence relations $\Theta_1, \ldots, \Theta_n$ of $\mathfrak{A}_{\mathscr{K}}^{(1)}$. Consider Θ_i as a congruence relation on $\mathfrak{A}_i$.

Take a congruence relations Θ of $\mathfrak{A}_{\mathscr{K}}^{(n)}$ having the following properties:

(5.1) the restriction of Θ to $\mathfrak{A}_i$ is $\geqslant \Theta_i$ $(i = 1, 2, \ldots, n)$;

(5.2) $\mathfrak{A}_{\mathscr{K}}^{(n)}/\Theta$ is isomorphic to a subalgebra of an algebra in $\mathscr{K}$.

If there exists a congruence relation which is the smallest one having properties (5.1) and (5.2), then it will be denoted by $\Sigma\Theta_i$.

5.3. *Let $a_1, \ldots, a_n \in A$, $(A; F) \in \mathscr{K}$. Then the following conditions are equivalent:*

(5.3.1) *$a_1, \ldots, a_n$ is a weakly independent sequence;*

(5.3.2) *let $b_1, \ldots, b_n \in B$, $(B; F) \in \mathscr{K}$, and $O(a_i) \leqslant O(b_i)$; then the mapping p: $a_i \to b_i$ $(i = 1, \ldots, n)$ can be extended to a homomorphism of $([a_1, \ldots, a_n]; F)$ into $(B; F)$;*

(5.3.3) *$\Sigma O(a_i)$ exists and*

$$\mathfrak{A}_{\mathscr{K}}^{(n)} / \Sigma O(a_i) \cong ([a_1, \ldots, a_n]; F), \qquad e_i^n / \Sigma O(a_i) \to a_i .$$

Suppose that $a_1, \ldots, a_n$ is weakly independent and the p of (5.3.2) is given. Define h as follows:

$$q(a_1, \ldots, a_n)h = q(b_1, \ldots, b_n) \quad \text{for every } q \in A^{(n)}.$$

Obviously, h maps $[a_1, \ldots, a_n]$ into $(B; F)$. This mapping is well-defined since $q_1(a_1, \ldots, a_n) = q_2(a_1, \ldots, a_n)$ $(q_1, q_2 \in A^{(n)})$ implies by 4.5 that $p(b_1, \ldots, b_n) = q(b_1, \ldots, b_n)$.

The mapping h is an extension of p since $a_i h = e_i^n(a_1, \ldots, a_n)h = e_i^n(b_1, \ldots, b_n) = b_i$.

Finally, h is a homomorphism. The proof of this is very similar to 3.6, so it can be omitted.

Thus (5.3.1) implies (5.3.2).

Next suppose that (5.3.2) holds and consider the mapping $e_i^n \to a_i$; this can be extended to a homomorphism h of $\mathfrak{A}_{\mathscr{K}}^{(n)}$ into $(A; F)$. Let Θ be the congruence relation induced by h. Then Θ satisfies (5.1) and (5.2). Indeed, if we restrict h to $\mathfrak{A}_i$, then we get a homomorphism of $\mathfrak{A}_i$ into $(A; F)$ carrying e_i^n into a_i. Since $\mathfrak{A}_i \cong \mathfrak{A}_{\mathscr{K}}^{(1)}$, we get that the congruence relation induced by the restriction of h on $\mathfrak{A}_i$ is $\geqq O(a_i)$. Thus (5.1) is verified; (5.2) is obvious. Now we prove that Θ is the smallest one satisfying (5.1) and (5.2). Indeed, if Φ satisfies (5.1) and (5.2), then consider $(B; F) \in \mathscr{K}$ of which $\mathfrak{A}^{(n)}_{\mathscr{K}}/\Phi$ is a subalgebra and let b_i denote the homomorphic image of e^n_i. Then by (5.1) $O(b_i) \geqq O(a_i)$, Thus by (5.3.2) the mapping p: $a_i \to b_i$ can be extended to a homomorphism k. Since the homomorphism which induces Φ equals the product hk, it follows that $\Theta \leqq \Phi$. Therefore $\Theta = \Sigma O(a_i)$ and we arrive at the isomorphism statement of (5.3.3).

Finally, suppose that (5.3.3) holds and let $p_1, p_2 \in A^{(n)}$, $p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n)$ and let $b_1, \ldots, b_n$ be given as in 4.5.

Let h_1 and h_2 be the homomorphisms induced by the mappings $e^n_i \to a_i$ and $e^n_i \to b_i$ $(n = 1, \ldots, n)$, respectively, and Θ_1, Θ_2 the congruence relation of $\mathfrak{A}^{(n)}_{\mathscr{K}}$ induced by h_1 and h_2 respectively.

Then by (5.5.3) $\Theta_1 = \Sigma O(a_i)$ and (4.5.3) imply that Θ_2 satisfies (5.1) and, obviously, it satisfies (5.2) as well. Hence by the definition of $\Sigma O(a_i)$ we get $\Theta_1 \leqq \Theta_2$.

By the same argument as in 3.6 we get that there exists a homomorphism

$$h\colon ([a_1, \ldots, a_n]; F) \to ([b_1, \ldots, b_n]; F)$$

such that

$$a_i h = b_i \qquad (i = 1, \ldots, h).$$

Therefore

$$\begin{aligned} p_1(b_1, \ldots, b_n) &= p_1(a_1 h, \ldots, a_n h) = p_1(a_1, \ldots, a_n) h \\ &= p_2(a_1, \ldots, a_n) h = p_2(a_1 h, \ldots, a_n h) = p_2(b_1, \ldots, b_n), \end{aligned}$$

which was to be proved.

The proof of 5.3 is completed.

The only difficult notion involved in 5.3 is that of $\Sigma\theta_i$. It should be remarked that in case $\mathscr{K}$ has special properties $\Sigma\theta_i$ can be more simply characterized.

5.4. *Suppose $\mathscr{K}$ contains the homomorphic images and subalgebras of algebras in $\mathscr{K}$. Then $\Sigma\Theta_i$ always exists.*

6. Consequences and problems.

6.1. *A single element a is always weakly independent.*

Proof. Use the characterization given by (5.3.3) and 3.6.

6.2. *The elements $a_1, \ldots, a_n$ of an abelian group are independent if and only if $\Sigma k_i a_i = 0$ implies $k_1 a_1 = \ldots = k_n a_n = 0$.*

In this case $\mathfrak{A}_{\mathscr{K}}^{(n)}$ is the free abelian group on n generators, thus (5.5.3) gives the isomorphism:

$$([a_1, \ldots, a_n]; +) = \sum ([a_i]; +),$$

which is equivalent to the statement of 6.2. The same result is true for an arbitrary module over a ring.

6.3. *In lattices independence and weakly independence is the same.*

This is true by 4.6.

6.4. *If $a_1, \ldots, a_n$ is a weakly independent sequence, then $a_{i_1}, \ldots, a_{i_n}$ is also weakly independent, where $i_1, \ldots, i_n$ is any permutation of $1, \ldots, n$.*

Thus we can speak of a weakly independent set I, which in the finite case means that any ordering of I gives a weakly independent sequence while in the infinite case it means that every finite subset of I is weakly independent.

A *basis* H of an algebra $(A; F) \in \mathscr{K}$ is a set which is weakly independent and generates $(A; F)$.

6.5. *To every integer n there corresponds a class $\mathscr{K}$ and an algebra $(A; F) \in \mathscr{K}$ such that $(A; F)$ has a basis of k elements if and only if $k \leqslant n$.*

Let $p_1, p_2, \ldots, p_n$ be distinct primes, $\mathfrak{C}_i = (C_i; +)$ be the cyclic group of order p_i, $\mathfrak{C} = \Sigma \mathfrak{C}_i$, $\mathscr{K} = \{\mathfrak{C}\}$. Then $\mathscr{K}$ is effective in 6.5.

One of the unpleasant surprises about weak independence is that it is possible that $a_1, \ldots, a_n$ be independent and $a_1 \in [a_2, \ldots, a_n]$ as it is shown by the following example:

6.6. Let $A = \{a_1, a_2\}$, $F = \{f, g\}$ and $f(a_1) = a_1$, $f(a_2) = a_1$; $g(a_1) = a_2$, $g(a_2) = a_2$, and $\mathscr{K} = \{(A, F)\}$. Then $\mathfrak{A}_{\mathscr{K}}^{(1)}$ consists of three elements x, y, z and $f(x) = f(y) = f(z) = z$ and $g(x) = g(y) = g(z) = y$. It is easy to see that $x \equiv z(O(a_1))$, while $x \not\equiv z(O(a_1))$ and $x \equiv y(O(a_2))$ while $x \not\equiv y(O(a_2))$. Hence neither $O(a_1) \leqslant O(a_2)$ nor $O(a_1) \geqslant O(a_2)$ hold. Therefore, the only mapping p satisfying the assumption of (5.3.2) is p: $a_1 \to a_1$, $a_2 \to a_2$, whence a_1, a_2 is an independent sequence and $a_1 \in [a_2]$, $a_2 \in [a_1]$.

Some of the problems, which arise very naturally, are the following:

Problem 1. Let $n_1, n_2, \ldots$ be a (finite or infinite) sequence of integers. Construct a class $\mathscr{K}$ and an algebra $(A; F) \in \mathscr{K}$ such that $(A; F)$ has a basis of k elements if and only if $k = n_i$ for some i (**P 602**).

Problem 2. Let A be a set and J a hereditary family of finite subsets of A including all one element subsets. Prove the existence of an algebra $(A; F)$ (with $\mathscr{K} = \{(A; F)\}$) such that a subset in $(A; F)$ is weakly independent if and only if it is contained in J (**P 603**).

Problem 3. Is it possible that an algebra without constant algebraic operations has a finite and also an infinite basis? Even if F is finite (**P 604**)?

Problem 4. Find sufficient conditions on the class $\mathscr{K}$ under which $a_1 \in [a_2, \ldots, a_n]$ implies that $a_1, \ldots, a_n$ is not independent and not all elements are torsion free (**P 605**).

Problem 5. Prove that if $\mathscr{K}$ is an equational class of algebras with anullary operation that determines a one-element subalgebra in every algebra in $\mathscr{K}$, and $a_1, \ldots, a_n$ is independent if and only if $[a_1, \ldots, a_n] = [a_1] \times \ldots \times [a_n]$, then $\mathscr{K}$ is equivalent to the class of all modules over a ring. (If this is not the case, what additional conditions are needed?) (**P 606**)

Problem 6. Work out the notion which corresponds to the notion of p-rank in Abelian groups (**P 607**).

Remark. The role of elements of order p should be taken by elements whose order $O(a)$ is a dual atom in $\mathfrak{C}(A_{\mathscr{K}}^{(1)}; F)$.

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PENNSYLVANIA STATE UNIVERSITY

Reçu par la Rédaction le 29. 9. 1964