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CLASS NUMBER AND FACTORIZATION IN QUADRATIC NUMBER FIELDS

BY

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0. The aim of this paper is to give a survey of results concerning factorizations into irreducible factors in a quadratic extension of the field Q of rational numbers, and the class number of such extension. We shall also quote some results which concern fields with larger degrees as far as they generalize the corresponding results for quadratic fields.

The list of references at the end of this paper contains also some papers which are not quoted in the text but which are related to our subject. Some of them were inaccessible to me neither in original nor in a review. They are indicated in the list by an asterisk.

The author would be grateful for any reference to a paper which is concerned with the subject of this survey but which is not included in this list.

For the definitions of notions occurring but not defined in the sequel see e.g. the books of Hasse [2] and Hecke [1]. The following notation will be preserved in the sequel: if K is an algebraic number field, then d is its discriminant (over Q) and n is its degree (also over Q). Moreover, H will denote the group of ideal classes of K, and H^* the group of ideal classes in the narrow sense (i.e. the group of fractional ideals modulo the group of all principal ideals generated by totally positive elements of the field; recall that a number $a \in K$ is said to be *totally positive* if all its real conjugates are positive; in particular, if K has no real conjugate fields, then all a in K are totally positive). Further, h and h^* denote the number of elements in H, resp. in H^*, i.e. the class number, resp. the "narrow" class number of K. (Note that the groups H and H^* coincide for quadratic K, except the case when d is positive and the fundamental unit of K has positive norm, in which case $h^* = 2h$). By $h(d)$, resp. $h^*(d)$ we shall denote the number h resp. h^* corresponding to the quadratic field with discriminant d. For quadratic K we shall denote by D the squarefree number whose quadratic root generates K.

All fields considered in the sequel are quadratic, unless the contrary is explicitly stated.

The group H^* can be also interpreted as the group of classes of binary quadratic forms with rational integral coefficients, of discriminant d, under composition. (For negative d only the positive definite forms should be considered). This fact can be used for the definition of H^* also for such integers N which are not field discriminants but are form discriminants. In particular, $h^*(N)$ obtains a meaning for such N's.

In § 1 we shall be concerned with asymptotic properties of the class number and the question when $h = 1$. The structure of H^* is discussed in § 2. Then the Euclidean fields are dealt with in § 3, whereas § 4 is devoted to other questions concerning our subject.

The importance of the class-group for number theory lies in the fact that it measures in some sense the deviation of multiplicative properties of K from those of a field whose ring of integers is a unique factorization domain (UFD). Clearly $h = 1$ if and only if the ring of integers of K is a UFD, and Carlitz [4] gave the following characterization of fields with $h \leqslant 2$:

(i) *For an arbitrary algebraic number field K the inequality $h \leqslant 2$ holds if and only if in every factorization of an integer from K into integers irreducible in K there occurs the same number of irreducible non-unit factors.*

A similar characterization of fields with a given class number exceeding 2 is not known.

1. The number h^* was extensively studied by many authors in the XIXth century starting with Gauss [1], who among others proved its finiteness. The literature of this period is surveyed by G. H. Cresse (in the third volume of Dickson [1]) who quotes 375 papers. The most important result of these times is due to Dirichlet, who gave explicit formulas for the class number:

$$(1)\qquad h = \begin{cases} \dfrac{w\sqrt{|d|}}{2\pi} L(1,\chi_d) = -\dfrac{w}{2|d|} \sideset{}{^+}\sum\limits_{x \bmod |d|} \left(\dfrac{d}{x}\right) x & (d<0), \\[2ex] \dfrac{\sqrt{d}}{2\log\varepsilon} L(1,\chi_d) = -\dfrac{1}{\log\varepsilon} \sideset{}{^+}\sum\limits_{\pm x \bmod d} \left(\dfrac{d}{x}\right) \log\left(2\sin\dfrac{\pi x}{d}\right) & (d>0), \end{cases}$$

where $\chi_d(m) = (d/m)$, $L(s,\chi_d) = \sum\limits_{m=1}^{\infty} \chi_d(m) m^{-s}$, w is the number of roots of unity contained in K, $\varepsilon > 1$ is the fundamental unit of K, $\sum\limits^{+}_{x \bmod |d|}$ denotes summation over the least positive full residue system mod $|d|$, and $\sum\limits^{+}_{\pm x \bmod d}$ denotes summation over the least positive residue half-system mod d.

These formulas are particular cases of similar formulas for the class number of fields with abelian Galois group over Q, which are essentially due to Kummer, who established them for cyclotomic fields $Q(\exp(2\pi i/m))$ (see Beeger [1], [2], Gut [1] and the book of Hasse [3]. For class number formulas for abelian extensions of quadratic fields see the book of Meyer [1]).

Usually (1) is proved by analytic means, but in the case of $d<0$, $d \not\equiv 1 (\mathrm{mod}\ 8)$ Venkov (Венков [1], [2], [3]) succeeded in giving an elementary proof.

Already Gauss conjectured that $h(d)$ tends to infinity when d tends to $-\infty$. E. Hecke (see the paper of Landau [3]) proved that this would follow from the extended Riemann's hypothesis. Assuming the falsity of Riemann's hypothesis, Deuring [1] showed that $\liminf_{-d\to\infty} h(d) \geqslant 2$, and, under the same assumption, Mordell [1] deduced Gauss's conjecture. Chowla [1] showed that if there are infinitely many negative d with $h(d)=1$, they must be very rare. The final step was done by Heilbronn [1] who proved the conjecture independently of all unproved assumptions. His result was made more precise by Landau [4] who proved the following result:

(ii) *There exists an absolute constant C such that for every m there is at most one field with $d< -Cm^8\log^6 m$ and $h(d)=m$.*

Heilbronn's result was improved by Siegel [1] who proved that for every positive ε and sufficiently large k

$$L(1,\chi) > k^{-\varepsilon}, \tag{2}$$

where χ is the real primitive character mod k, which result in turn implies

$$\lim \frac{h(d)}{\log|d|} = \frac{1}{2}. \tag{3}$$

Other proofs of this important result were given by Heilbronn [2], Čudakov (Чудаков [1]), Linnik (Линник [2]), Estermann [1] and Chowla [8]. An elementary proof based on the method of Vinogradov was found by Linnik (Линник [4]), whose paper contains also a new analytic proof.

Siegel's theorem was generalized to arbitrary fields by Brauer [1], [2] who proved that

(iii) *If $K_1, K_2, \ldots$ is a sequence of normal algebraic number fields with degrees $n_1, n_2, \ldots$ and discriminants $d_1, d_2, \ldots$ which satisfy the relation $n_k = o(\log|d_k|)$, then*

$$\lim_{k\to\infty} \big(\log(R_k h_k)\big)/\log|d_k| = 1/2, \tag{4}$$

where R_k is the regulator, and h_k the class number of K_k.

Relation (4) was subsequently improved by Vinogradov (Виноградов [1], [2]) to the form

$$\log(hR) = \log|d|^{1/2} + \log(1-\beta) + O(\log\log|d|) \tag{5}$$

holding for fields of fixed degree and $|d|$ tending to infinity. Here β is the possible exceptional zero of the function $\zeta_K(s)$ (see also paper [3] of the same author, where it is shown that the term $\log(1-\beta)$ can be neglected if a set of exceptional fields is excluded).

As the value of the regulator is, in general, unknown, neither (4) nor (5) give much information about h. However Ankeny, Brauer and Chowla in a joint paper [1] succeeded in proving the following result:

(iv) *Given a positive ε, there exist infinitely many fields of given degree (and even of given signature) with $h > |d|^{1/2-\varepsilon}$.*

(Note that for any field of degree n, $h = O(|d|^{1/2}(\log|d|)^{n-1})$, as shown in Landau [1]).

For a result similar to (3) concerning the ratio of the class numbers of the cyclotomic field $Q(\exp(2\pi i/p))$ (where p is a rational prime) and of its maximal real subfield, see Ankeny, Chowla [1], [3] and Siegel [2].

Landau's result quoted above can be made more precise for quadratic fields. For imaginary quadratic fields (with $d \neq -3$) $h(d)$ does not exceed $(2\pi)^{-1}\log|e^2 d|\,|d|^{1/2}$, whereas for real quadratic fields the inequality $h(d) < d^{1/2}$ holds for $d > 4$ (see Slavutsky (Славутский [3]) and Newman [1]. Weaker results are contained in Ankeny, Chowla [5], Carlitz [3] and Gut [4]). Formula (1) shows that asymptotic properties of h for negative d are determined by those of $L(1, \chi_d)$. It is conjectured that $L(1, \chi_d) \geqslant C(\log d)^{-1}$ with some positive constant C. E. Hecke (see Landau [3], Mahler [1]) showed that the extended Riemann's hypothesis implies this conjecture.

Littlewood [1] showed under the same assumption that, for infinitely many k, $L(1, \chi_k) \geqslant C\log\log k$ and, for infinitely many k, $L(1, \chi_k) \leqslant C_1(\log\log k)^{-1}$ hold with some positive C and C_1. Chowla [4] proved the first of these results unconditionally and the second was proved by Walfisz [1] and Linnik (Линник [1]). Walfisz showed also that one can put $C = \exp\gamma$, where γ is Euler's constant (for a simpler proof see Chowla [6]) and Chowla [7] gave a numerical value for C_1 (see also Chowla [5]). An interesting result was obtained by Fluch [1]:

(v) *There exists an absolute constant M such that for every $k > 2$ and every non-principal character χ $\mathrm{mod}\, k$ either $L(1, \chi) \geqslant M(\log k)^{-1}$ or $L'(1, \chi) > 1$.*

The mean value of $h^*(d)$ for negative d was investigated by Vinogradov (Виноградов [1], [2], [3], [4]; see also Fomenko [1]). The best evaluation of the remainder-term is contained in [4]:

$$\sum_{-N \leqslant d < 0} h^*(d) = \frac{4\pi}{21\,\zeta(3)} N^{3/2} - \frac{2}{\pi} N + O(N^{2/3+\varepsilon}). \tag{6}$$

The sums $\sum_{-N\leqslant d<0}(h^*(d))^k$ were investigated by Lavrik (Лаврик [1]) for $k=2,3$, Saparnijazov (Сапарниязов [1]) for $k=4,5$ and Barban (Барбан [1]) for arbitrary k. The best evaluation of the remainder-term is due to Barban and Gordover (Барбан, Гордовер [1]) who proved

$$(7)\qquad \sum_{-N\leqslant d<0}(h^*(d))^k = \frac{2^{1+k}r(k)}{\pi^k(k+2)}N^{1+k/2}+O(N^{\beta})\ (^1),$$

where $\beta=(k+2)/2\big(1-(5k^2)^{-1}\big)$, $r(k)=\sum_{j\equiv 1(\mathrm{mod}\,2)}\varphi(j)\tau_k(j^2)j^{-3}$ and $\tau_k(m)$ is the number of solutions of the equation $x_1\dots x_k=m$ in natural numbers.

Now let us turn to the question for which d we have $h(d)=1$. The problem whether there exist infinitely many such quadratic fields is open (clearly for imaginary fields the answer is negative in view of (3)). It was Gauss who proved that if t is the number of distinct prime divisors of d, then 2^{t-1} divides $h^*(d)$, and in fact the group H^* has $t-1$ even invariants. (If an abelian group G is factorized into cyclic groups $C_{p_1^{a_1}},\dots,C_{p_r^{a_r}}$, where $p_1,\dots,p_r$ are primes, then the numbers $p_i^{a_i}$ $(i=1,2,\dots,r)$ are called the *invariants* of G). It results that $h^*(d)$ is odd if and only if d is a prime power, i.e. if $d=-4,\pm 8$, or $d=\pm p$, a prime congruent to ± 1 (mod 4). A similar result concerning cyclic extensions of prime degree is contained in paper [1] of Leopoldt (see also Fröhlich [1], where a similar, although weaker result is proved for abelian extensions of Q).

Ennola [2] gave a simple elementary proof of the following result:

(vi) *If $h=1$, then taking aside the case $D=-1$, $|D|$ must be either a prime or a product of two primes congruent to unity* (mod 4). *If, moreover, D is negative, then $|D|$ must be a prime incongruent to unity* (mod 4).

(For positive D, a simple proof was supplied by Inkeri [2]. See also Popovici [2], Rédei [15], Rodica [1] and Rusu [1].)

For negative d, various authors have given necessary and sufficient conditions for $h=1$. Let us quote some of them:

(vii) *If $d=1-4m<0$, then $h(d)=1$ if and only if all numbers x^2-x+m $(x=1,2,\dots,m-1)$ are primes* (Rabinowitsch [1]. See also Connell [1], Frobenius [1] and Mařik [1]).

(viii) *If $D<-7$, then $h=1$ if and only if $-D$ is a prime congruent to* 3 (mod 8) *and for all rational primes p not exceeding $((p+16)/3)^{1/2}-2$ the equality $(D/p)=-1$ holds* (Nagell [1]).

Nine imaginary quadratic fields with $h=1$ are now known, namely those with $D=-1,-2,-3,-7,-11,-19,-43,-67$ and -163

(1) *Added in proof.* Recently Barban (ДАН СССР 172 (1967), p. 999-1000) improved the evaluation of the remainder term in (7) to $O(N^{1+k/2-a})$ with any $a<(\sqrt{129}-9)/2$.

(see Lánczi [1] for a unified proof of this fact. For $D = -1$ see e.g. Popovici [1] and Rudin [1]). As proved in Heilbronn, Linfoot [1] there can exist at most one more such field. Lehmer [1] showed that for such a field one would have $|d| \geqslant 5\cdot 10^9$ and recently Stark [1] improved this bound to $|d| > \exp(22\cdot 10^6)$. The truth of the extended Riemann's hypothesis would imply the non-existence of the tenth such field (recently Stark proved the non-existence of such a field). In fact even less will do, as demonstrated by Bateman and Grosswald [1] who proved that

(ix) *If p is a prime larger than* 163 *and $h(-p) = 1$, then the function $L(s, \chi_p)$ has a real zero x satisfying*

$$1 - \frac{6}{\pi\sqrt{p}}\left(1 + \frac{6\log\frac{p}{4}}{\pi\sqrt{p}}\right) \leqslant x < 1.$$

They quote an unpublished result of J. B. Rosser showing that in the interval $(1-6\pi^{-1}p^{-1/2}, 1)$ no such x can exist. A previous result of Chowla and Selberg [1] gave a larger interval for x, namely $(\frac{1}{2}, 1)$. It is known (Rosser [1]) that for d which are discriminants and satisfy the inequality $-227 < d < 0$ with the possible exception of $d = -163$, $L(s, \chi_d)$ is positive for positive values of s (cf. also Chowla, Erdös [1] and Chowla, Selberg [1]). This result was recently extended by Low [1] for discriminants d in the range between 0 and $-593\,000$ (exclusive, with the possible exception of $d = -115147$). A simple proof of not vanishing of $L(1, \chi_p)$ was given in Chowla and Mordell [1] (see also Stoll [1]).

An old question whether every field can be imbedded in a field with $h = 1$ was reduced by Šafarevič (Шафаревич [1]) to a group theoretical problem which found a negative solution in the joint paper of Golod and Šafarevič (Голод, Шафаревич [1]). As a corollary of their main result they proved that if d is negative and has at least 7 distinct prime divisors, then the field $Q(d^{1/2})$ cannot be imbedded in such a way, and in fact it has an infinite class field tower.

2. As we said above, the number of even invariants of H^* is equal to $t-1$, where $t = t(d)$ is the number of distinct primes dividing d. Rédei and Reichardt [1] found a similar formula for the number e_4 of invariants of H^* divisible by 4:

(x) *If $F(d)$ is the number of factorizations $d = d_1 d_2$, where both d_1 and d_2 are discriminants, and for every prime p dividing d_1 as well as for every prime q dividing d_2 one has*

$$\left(\frac{d_1}{q}\right) = \left(\frac{d_2}{p}\right) = 1,$$

then $F(d) = 2^{e_4}$.

Rédei [5] proved also that $e_4 \leqslant r_1+r_2-1$, where r_1 is the number of positive factors in the factorization of d into relatively prime discriminants, and $2r_2$ is equal either to the number S of negative factors if it is even, or to $S+1$ otherwise.

The number e_{2^n} of invariants of H^* divisible by 2^n was described in a way similar to (x) by Reichardt [1]: a factorization $d = d_1 d_2$ is called a *decomposition of the type* N if d_i are both discriminants and there exists a field K containing $d_1^{1/2}$ and $d_2^{1/2}$, unramified (i.e. with the relative discriminant equal to the unit ideal) and cyclic of the degree 2^{N-1} over $Q(d^{1/2})$, in which all prime divisors of d decompose completely. All such decompositions (where the decompositions $d = d_1 d_2$ and $d = d_2 d_1$ are not distinguished, and the trivial decomposition $d = 1 \cdot d$ is included) form an abelian group under multiplication defined by

$$\{d_1, d_2\} \cdot \{D_1, D_2\} = \{A_1, A_2\},$$

where A_i equals the product $d_i D_1$ divided by a suitable quadratic factor chosen in such a way that one gets discriminants after division by it. With this notation Reichardt proved that

(xi) *The number* e_{2^N} *is equal to the maximal number of independent elements in the group of decompositions of the type* N, *which turns out to be a product of cyclic groups of order two.*

(See also Rédei [6], [8], [14]. In the second of those papers the existence of infinitely many quadratic fields with given $e_2 \geqslant e_4 \geqslant e_8 \geqslant 0$ is proved. See also Iyanaga [1]).

The more general problem of determination of the number of invariants of H^* divisible by a given prime power was solved by Rédei [12], [13] for arbitrary, not necessarily quadratic fields. A special case was also treated by Inaba [1]. The results, however, as should be expected, are much more complicated than (xi).

The probability that for a quadratic field the number e_4 assumes a given value was found by Rédei [7], whose paper contains also results concerning the mean value of e_4.

The residue of $h(d)$ mod $2^{t(d)}$ was found for negative d by Rédei [1]. (The cases $t(d) = 2, 3$ were solved by Hurwitz [1]. Plancherel [1] studied the general case and gave a recurrent formula for the solution of this problem, but he did not give explicit results except $t = 4, 5$ (see also Lerch [1]). Cf. also a recent paper of Pumplün [1], where the same problem is considered.

Chowla [2] proved that the ratio $h(d)/2^{t(d)-1}$ tends to infinity for d tending to $-\infty$ and, as $t(d)-1$ is the number of even invariants of H (note that for negative d, H and H^* coincide), there exist only finitely many negative discriminants such that the corresponding class-group has the form $C_2 \times C_2 \times \ldots \times C_2$ (see also Chowla, Briggs [1], Grosswald [1], Hall [1], Swift [1] and Papkov (Папков [4])).

It is not known whether every finite abelian group can serve as H or H^* for a suitable quadratic field. The result of S. Chowla quoted above shows that some groups cannot be equal to H for imaginary quadratic fields. It is not known whether every natural number can serve as the class-number for a suitable imaginary quadratic number field. However, it was proved by Nagell [1] that there exist infinitely many imaginary quadratic fields with the class-number divisible by a given number (see also Ankeny, Chowla [4] and Humbert [1]). This result was improved by Kuroda [1], who showed that one can find such fields even with the discriminant divisible by a given squarefree number.

For cyclotomic fields, Fröhlich [2] obtained a result analogous to that of Nagell (Cf. also Gut [3] for partial results on a similar problem for relatively-quadratic extensions).

Gauss called a discriminant *regular* if the corresponding group H^* is cyclic. In Dickson [1] (Ch. V, vol. III) the older literature on irregular discriminants is surveyed. Not much has been added to this theory since then (see e.g. Pall [1] and Lippman [1]).

For various conjectures related to the structure of H^* cf. Šafarevič (Шафаревич [2]). In particular, the following question is unsolved: does there exist a constant C such that the group H^* has at most C generators, provided the discriminant is a prime?

3. A field $K = Q(\vartheta)$ is said to be *euclidean* if for every pair of non-zero integers a, b in K there exists an integer c in K such that $|N(a-bc)| < |N(b)|$ (where $N(x)$ is the norm of $x \in K$) or, which means the same, if for every number a in K there is an integer b in K with $|N(a-b)| < 1$. Every euclidean field has a trivial class group, but already R. Dedekind observed that the converse implication fails, as the field $Q((-19)^{1/2})$ is not euclidean, whereas $h = 1$. It is easy to show, and was first observed by Dickson [2] (p. 150-151) that among the imaginary quadratic fields there is only a finite number of euclidean ones, namely those with discriminants $-3, -4, -7, -8$ and -11. Dickson asserted moreover that among the real quadratic fields only those with discriminants 5, 8, 12 and 13 are euclidean, but his proof was incorrect, and in fact Perron [1] found six more euclidean fields (with $d = 17, 21, 24, 28, 29$ and 44). Later on it was found that the fields with $d = 33, 37, 41, 57, 73$ and 76 are also euclidean (see Oppenheim [1], Remak [1] and Rédei [11]).

Heilbronn [3] succeeded in 1938 in proving that the number of euclidean real quadratic fields is finite (a little earlier Erdős and Chao Ko [1] got this result for the fields generated by the square root of a prime). It turned out later that there are only those listed above. The final step in proving this was done by Chatland and Davenport [1] and was based on results of numerous authors who treated several special cases: so Berg [1] showed that if the field $Q(D^{1/2})$ is euclidean and $D \equiv 2, 3 \pmod 4$, then D does not exceed 100. Hofreiter [2] proved that there are no euclidean fields with $d \equiv 14 \pmod{24}$. Behrbohm and Rédei [1] found all euclidean fields with $D \equiv 2, 3 \pmod 4$ and $D \equiv 5 \pmod{24}$. Loo-Keng Hua [2] proved that if $D \equiv 1 \pmod 4$ and the field is euclidean, then D does not exceed e^{250}, reducing thus the problem to the study of a finite number of fields. Davenport [4] improved this, showing that the discriminant of a euclidean field does not exceed 2^{14} (see also Brauer [1], Chatland [1] (this paper contains a historical survey), Cugiani [1], Fox [1], Hofreiter [1], Loo-Keng Hua and Min [1], Loo-Keng Hua and Shih [1], Inkeri [1], [2], Rédei [9], [10], Schuster [1], Sudan [1], Yang [1]). Note also that Barnes and Swinnerton-Dyer [1] proved that $Q\left((97)^{1/2}\right)$ is not euclidean, correcting thus a misstatement in Rédei [11].

Varnavides [1] proved by a unified method that all fields listed above are euclidean (see also Ennola [1]).

Motzkin [1] considered a general situation of a domain R with a norm $|x|$ satisfying the following postulates:

(a) $|x|$ is a positive rational integer for non-zero x,

(b) $|x| \leqslant |xy|$ holds for $x, y \neq 0$, and

(c) For every $b \neq 0$ and a not divisible by b there exists a $q \in R$ such that $|a-qb| < |b|$.

He proved that in the ring of all integers of an imaginary quadratic field such a norm exists only in the classical cases, i.e. for $d = -3, -4, -7, -8$ and -11. This gave a negative answer to a question of Hasse [1]. Essentially the same result was later rediscovered by Dubois and Steger [1] who proved moreover that in those cases $|x|$ must coincide with $N(x)$. A similar problem for real quadratic fields seems to be unsolved.

Euclidean fields which are not quadratic were considered too. Davenport [1], [2] proved that every euclidean cubic field with negative discriminant must satisfy $|d| \leqslant 64 \cdot 10^{26}$ and he obtained in [3] a similar result for totally imaginary quartic fields, the bound in this case being $(4 \cdot 10^8 \cdot A^3)^8$ with $A = 16 \cdot 10^8 + 2$. Heilbronn [4] proved the finiteness of the set of euclidean cubic normal fields and conjectured that this is not true for non-normal cubic fields. This conjecture is not settled yet (see also Heilbronn [5]).

Clarke [1] showed that the non-cyclic cubic field generated by a solution of $x^3 - 4x + 2 = 0$ (which has $d = 148$) is euclidean. It is known also that the totally real cubic fields with d not exceeding 64 as well as those with $d = 81, 169, 229, 257, 316, 321$ and 361 are euclidean (see Godwin [1], who extended previous results of H. Davenport). The following totally quadratic real fields are known to be euclidean: $d = 725$, $1125, 1600, 1957, 2304, 2624, 2777$ and 4205 (see Godwin [2]). In the same paper H. J. Godwin showed that the field $Q(2\cos\pi/11)$, which is the real quintic field of the least discriminant, is euclidean. For the fields with larger degrees nothing is known concerning their euclidicity, except the case of cyclic fields, where some results were obtained by Heilbronn [5].

4. There are interesting connections between the class number of quadratic fields and the numbers of Bernoulli. Some of them were known already to Cauchy, as the following one:

(xii) *If p is a prime, $p \equiv 7 \pmod 8$, then*

$$h(-p) \equiv 2B_{(p+1)/2} \pmod p$$

and if p is a prime, $p \equiv 3 \pmod 8$, then

$$h(-p) \equiv -6B_{(p+1)/2} \pmod p.$$

(See Dickson [1], vol. III. p. 102. Cf. also the important paper of Hurwitz [1], where many other similar congruences for h were proved.)

The numbers B_j occurring here are defined by

$$t(e^t-1)^{-1} = \sum_{m=0}^{\infty} B_m t^m/(m!).$$

The following result obtained by Kiselev (Киселев [1], [2]) and independently by Ankeny, Artin and Chowla [1], [2], gives a connection between the class number, fundamental unit and the Bernoulli numbers in the case of a real quadratic field:

(xiii) *If $p \equiv 1 \pmod 4$ is a prime and $\varepsilon = T + Up^{1/2} > 1$ is the fundamental unit of the field $Q(p^{1/2})$, then*

$$T \equiv (-1)^{(1+h)/2}[(p-1)/2]! \pmod p$$

and

$$hU \equiv TB_{(p-1)/2} \pmod p.$$

(See also S. Chowla [10].)

If χ is a character mod f, then the generalized Bernoulli numbers belonging to χ are defined by the formula

$$\sum_{r=1}^{f} \frac{\chi(r)te^{rt}}{e^{tf}-1} = \sum_{k=0}^{\infty} B_\chi^{(k)} t^k/k! \qquad \left(|t| < \frac{2\pi}{f}\right).$$

(See Berger [1] and Leopoldt [2] for properties of those numbers.)

Slavutsky (Славутский [4]) proved that

(xiv) *If* $d = mp$, $p \geqslant 5$ *is a prime*, $1 \leqslant m < p$ *and* $T+Ud^{1/2} = \varepsilon > 1$ *is the fundamental unit of* $Q(d^{1/2})$, *then* p^k *divides* U *if and only if* p^k *divides* $j^{-1}B_\chi^{(j)}$, *where*

$$\chi(x) = \left(\frac{(-1)^{(p-1)/2}m}{x}\right) \quad \textit{and} \quad j = (p-1)p^{k-1}/2.$$

(This generalizes (xiii) as h occurring there can never be divisible by p. See e.g. Ankeny and Chowla [5] and Slavutsky (Славутский [5])).

For other results concerning h, ε and the Bernoulli numbers see Ankeny and Chowla [6], Chowla [9], Carlitz [1], [2], [3], Kiselev and Slavutsky (Киселев, Славутский [1], [2]), Mordell [2], and Slavutsky (Славутский [1], [2], [5]) (Cf. also Teege [1] and Mordell [4]).

Leopoldt [3] found a connection between the generalized Bernoulli numbers and the class number of real abelian fields (generalizing thus a result of Kummer concerning maximal real subfields of cyclotomic fields) and, as an application, he obtained the following result concerning real quadratic fields:

(xv) *If* p *is a prime*, $p \equiv 1 \pmod 4$ *and* $\chi(x) = (x/p)$, *then either there exists an odd prime* q *less than* $\frac{1}{2}p^{1/2}\log p$ *such that the number* $B_\chi^{(q-1)}$ *is divisible by* q *or* $h(p) = 1$.

The quantitative approach to the problem of factorization in quadratic fields was apparently initiated by Fogels [1], who proved that in the field $Q\left((-5)^{1/2}\right)$ (which has $h = 2$) almost none rational integer has a unique factorization into irreducible factors, and that a similar result holds for integers of this field. (One says that *almost none integer of the field* K *has a property*, say, P if the number $M(x)$ of non-associated integers in K with the property P and with norms not exceeding x in absolute value, satisfies $M(x) = o(x)$ as x tends to infinity).

These results were generalized in Narkiewicz [1] to all normal fields (and the second of them to all algebraic fields) with $h \neq 1$. An analogous result concerning rational integers, resp. the integers of the field having all factorizations of the same length (or, more generally, having at most M, a given number of factorizations of different lengths) was obtained in Narkiewicz [1], [2] for fields with $h \neq 1, 2$ (the result concerning rational integers, under the only restriction that the field is normal). The case of quadratic fields was more closely examined in Narkiewicz [3], [4], where the asymptotic behavior of the counting function of rational integers with unique factorization (or with all factorizations of the same length) was found.

If these functions are denoted by $F(x)$ and $G(x)$, then

$$\begin{aligned} F(x) &\sim C_1 x(\log\log x)^M(\log x)^{(1-h)/2h}, \\ G(x) &\sim C_2 x(\log\log x)^N(\log x)^{(1+g-h)/2h}, \end{aligned} \tag{8}$$

where C_1 and C_2 are positive constants depending on the field, g is the number of even invariants of the class group H, and M and N are non-negative rational integers, depending on H, which are described in Narkiewicz [3], [4]. If H is cyclic, then $M = [\log h/\log 2]$ and N is equal to $h-1$ in the case of odd h, and is equal to $(h-2)/2$ in the case of even h.

In the same paper Narkiewicz [4] the distribution of rational integers with unique factorization (resp. with all factorizations of the same length) in residue classes (mod k) was examined, with k relatively prime to the discriminant. (Note that in Narkiewicz [3] the condition on k was omitted, and in fact, it turned out later (see Narkiewicz [6]), that it is not necessary for the validity of results in Narkiewicz [4]).

The following problem seems to be unsolved: If K is a quadratic field with $h \neq 1$, do there exist infinitely many pairs $n, n+1$ of consecutive integers, both of them having unique factorization in K? The same question can be stated for arbitrary fields with $h \neq 1$ in which the number 2 has a unique factorization. Similar question can be asked concerning numbers with unique factorization occurring in polynomial sequences $\{P(n)\}$, under obvious restrictions on the polynomial $P(x)$.

The mean value of the number of different factorizations of integers in a given field was found by Rémond [1] (for some other results concerning this number see Narkiewicz [5]).

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