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REMARKS ON ALGEBRAS AND REGULAR FINITE PLANES

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Let us recall the notion of an A^k-algebra, introduced in [2].

Definition. An *A^k-algebra* is an ordered pair $\langle A, \circ\rangle$, where A is a non-empty set of arbitrary elements and $\circ$ is a binary operation $\circ: A^2 \to A$ fulfilling the following system of k axioms:

(W_0) $x \circ x = x$,

(W_i) $x^i[y] = y^{\varphi(i)}[x]$ for $i = 1, 2, \ldots, k-2$,

(W_{k-1}) $x^{k-1}[y] = y$,

where $x^1[y] = x \circ y$, $x^{n+1}[y] = x \circ (x^n[y])$ and $\varphi: \{1, 2, \ldots, k-2\} \to \{1, 2, \ldots, k-2\}$ is a fixed function.

It has been proved in [2] that every A^k-algebra $\mathfrak{A} = \langle A, \circ\rangle$ which satisfies axioms (W_0)-(W_{k-1}) is a quasi-group and that each pair of different elements of A generates a subset of A containing exactly k elements. Moreover, the existence of a non-contradictory system of axioms (W_0)-(W_{k-1}) is ensured for each $k = p^m$ if p is a prime number.

We say that the elements generated by a pair a, b ($a \neq b$) are *collinear*. An A^k-algebra A for which every triple of non-collinear elements of A generates exactly k^2 different elements (the affine essential plane as in [1]) is called an *A_2A^k-algebra*. Each A_2A^k-algebra which is an essential plane is an affine plane ($A^k_{k_2}$-algebra as in [1]).

In this paper some properties of A_2A^k-algebras are considered and some new problems concerning such algebras are formulated.

1. THEOREM 1. *If an A^k-algebra fulfills the condition $x \circ (y \circ z) = z \circ (y \circ x)$, then $\varphi(1) = 2$* [1].

To prove Theorem 1 it is enough to put $x = y$ and to apply the condition (W_0).

I am going to prove

THEOREM 2. *Each A^k-algebra $\mathfrak{A}$ containing at least $k+1$ elements for which $\varphi(1) = 2$ and $x \circ (y \circ z) = z \circ (y \circ x)$ is an A_2A^k-algebra.*

[1] Algebras A^4, $A^{5''}$ and $A^{9'}$ in notation of [2] may serve as examples of A^k-algebras.

Proof. By the suppositions of theorem, $\mathfrak{A}$ fulfills the following axioms:

(W_0) $x[x] = x$,

(W_1) $x[y] = y^2[x]$,

(W_i) $x^i[y] = y^{\varphi(i)}[x]$ for $i = 3, 4, \ldots, k-2$; $\varphi(i) \in \{3, 4, \ldots, k-2\}$,

(W_{k-1}) $x^{k-1}[y] = y$,

(W_k) $x \circ (y \circ z) = z \circ (y \circ x)$.

Let $a_1, a_2 \in A$ and $a_1 \neq a_2$. Then a_1, a_2 generate a k-elements set $\{a_1, \ldots, a_k\}$. Let $b_0 \in A$ and $b_0 \notin \{a_1, \ldots, a_k\}$. Denote by $b_{i,0} = a_i \circ b_0$, $b_{i,1} = a_i \circ b_{1,0}$, $b_{i,2} = a_i \circ b_{1,1}, \ldots, b_{i,k-2} = a_i \circ b_{1,k-3}$.

It is evident that $b_{1,k-2} = a_1 \circ b_{1,k-3} = a_1 \circ (a_1 \circ b_{1,k-4}) = \ldots = a_1^{k-1}[b_0] = b_0$.

Let us take $\bar{A} = \{a_1, \ldots, a_k, b_0, b_{1,0}, \ldots, b_{k,0}, b_{11}, \ldots, b_{k,1}, b_{1,k-3}, \ldots \ldots, b_{k,k-3}, b_{2,k-2}, \ldots, b_{k,k-2}\}$.

We shall prove first that the product of each pair of elements of $\bar{A}$ belongs to $\bar{A}$.

Let $a_i \circ a_p = a_1 \circ a_m$ and $a_1 \circ a_i = a_s$. Then $a_p \circ b_{i,j} = a_p \circ (a_i \circ b_{1,j-1}) = b_{1,j-1} \circ (a_i \circ a_p) = b_{1,j-1} \circ (a_1 \circ a_m) = a_m \circ (a_1 \circ b_{1,j-1}) = a_m \circ b_{1,j} = b_{m,j+1} \in \bar{A}$.

We have $a_p \circ b_{1,k-2} = a_p \circ (a_i \circ b_{1,k-3}) = b_{1,k-3} \circ (a_i \circ a_p) = b_{1,k-3} \circ (a_1 \circ a_m) = a_m \circ (a_1 \circ b_{1,k-3}) = a_m \circ b_{1,k-2} = a_m \circ b_0 = b_{m,0}$.

We are going to prove that $b_0 \circ b_{1,i} = b_0 \circ (a_1 \circ b_{1,i-1}) = \ldots = b_0 \circ (a_1^{i+1}[b_0]) = b_0 \circ (b_0^{\varphi(i+1)}[a_1]) = b_0^{\varphi(i+1)+1}[a_1] = a_1^{\varphi[\varphi(i+1)+1]}[b_0] = b_{1,t} \in \bar{A}$.

Then $b_{i,j} \circ a_k = a_k^2[b_{i,j}] \in \bar{A}$, and $b_{1,i} \circ b_0 = b_0^2[b_{1,i}] = b_0 \circ (b_0 \circ b_{1,i}) = b_0 \circ b_{1,t} \in \bar{A}$.

On the other hand, we have $b_0 \circ b_{i,j} = b_0 \circ (a_i \circ b_{1,j-1}) = b_0 \circ \big(a_i \circ (a_1 \circ b_{1,j-2})\big) = b_0 \circ \big(b_{1,j-2} \circ (a_1 \circ a_i)\big) = b_0 \circ (b_{1,j-2} \circ a_s) = a_s \circ (b_{1,j-2} \circ b_0) \in \bar{A}$, $b_{1,i} \circ b_{1,j} = b_{1,i} \circ (a_1 \circ b_{1,j-1}) = b_{1,j-1} \circ (a_1 \circ b_{1,i}) = b_{1,j-1} \circ (a_1^2[b_{1,i-1}]) = b_{1,j-1} \circ (b_{1,i-1} \circ a_1) = a_1 \circ (b_{1,i-1} \circ b_{1,j-1})$.

If we decrease the coefficients i, j repeatidly as above, we come to the conclusion that $b_{1,i} \circ b_{1,j} = a_1^{i+1}[b_0 \circ b_{1,t}] \in \bar{A}$.

Furthermore, $b_{1,r} \circ b_{i,j} = b_{1,r} \circ (a_1 \circ b_{1,j-1}) = b_{1,r} \circ \big(a_i \circ (a_1 \circ b_{1,j-2})\big) = b_{1,r} \circ \big(b_{1,j-2} \circ (a_1 \circ a_i)\big) = b_{1,r} \circ (b_{1,j-2} \circ a_s) = a_s \circ (b_{1,j-2} \circ b_{1,r}) \in \bar{A}$.

Finally, $b_{r,u} \circ b_{i,j} = b_{r,u} \circ (a_i \circ b_{1,j-1}) = b_{r,u} \circ \big(a_i \circ (a_1 \circ b_{1,j-2})\big) = b_{r,u} \circ \big(b_{1,j-2} \circ (a_1 \circ a_i)\big) = b_{r,u} \circ (b_{1,j-2} \circ a_s) = a_s \circ (b_{1,j-2} \circ b_{r,u}) \in \bar{A}$.

We have thus proved that the elements generated by any non-collinear elements a_1, a_2, b_0 of $\bar{A}$ belong to $\bar{A}$. Now we shall prove that $\bar{A}$ contains exactly k^2 elements or that all elements of the set $\bar{A}$ are different.

We shall prove first

L1. $b_{i,s} \neq a_j$.

Let us suppose that $b_{i,s} = a_j$. Then $a_i \circ (a_1^s[b_0]) = a_j$, $a_1^s[b_0] = a_i^{k-2}[a_j]$ or $b_0 = a_1^{k-1-s}\big[a_i^{k-2}[a_j]\big] = a_t$, which contradicts the supposition $b_0 \notin \{a_1, \ldots, a_k\}$.

L2. *For each* $b_{m,n}$ *and for each* t *there exists an* s *such that* $a_s \circ b_{m,n} = b_{t,n+1}$.

Let $a_s = a_m^{k-2}[a_1 \circ a_t]$, whence $a_m \circ a_s = a_1 \circ a_t$. Then $a_s \circ b_{m,n} = a_s \circ (a_m \circ b_{1,n}) = b_{1,n} \circ (a_m \circ a_s) = b_{1,n} \circ (a_1 \circ a_t) = a_t \circ (a_1 \circ b_{1,n}) = b_{t,n+1}$.

L3. *Let* $a_1 \circ a_i = a_p$, *whence* $a_i \circ a_1 = a_1 \circ a_p$. *Then* $a_1 \circ b_{i,j} = a_1 \circ (a_i \circ b_{1,j-1}) = b_{1,j-1} \circ (a_i \circ a_1) = b_{1,j-1} \circ (a_1 \circ a_p) = a_p \circ (a_1 \circ b_{1,j-1}) = a_p \circ b_{1,j} = b_{p,j+1}$.

From L2 and L3 we immediately obtain

L4. *For each* $b_{i,j}$ *and* $b_{m,n}$ $(j < n)$ *there exists an* s *such that* $b_{i,j} = a_1^{n-j+1}[a_s \circ b_{m,n}]$.

Now we shall prove that

(*) *For every* $b_{i,j}$ *and* $b_{m,n}$ $(j < n)$ *(it is clear that* $n-j \leqslant k-2$*) we have* $b_{i,j} \neq b_{m,n}$.

Suppose that $b_{i,j} = b_{m,n}$. Taking L4 into account we could obtain $a_1^{n-j+1}[a_s \circ b_{m,n}] = b_{m,n}$, then $a_s \circ b_{m,n} = a_1^{k-n+j-2}[b_{m,n}]$ and then $b_{m,n}^2[a_s] = b_{m,n}^{\varphi(k-n+j-2)}[a_1]$. The above equation gives $a_s = b_{m,n}^{\varphi(k-n+j-2)-2}[a_1] = a_1^{\varphi(\varphi(k-n+j-2)-2)}[b_{m,n}]$, which leads to a contradiction (see L1 and L3).

We shall prove also that

(**) $b_{i,s} \neq b_{j,s}$ *for* $i \neq j$.

Let us suppose that $b_{i,s} = b_{j,s}$ $(i \neq j)$. We have $a_i \circ b_{1,s-1} = a_j \circ b_{1,s-1}$, whence $a_i = a_j$ which leads to a contradiction.

From L1, (*) and (**) it immediately results that the set $\bar{A}$ contains exactly k^2 elements.

2. It is easy to prove that

(1) A^3-algebras fulfilling the axiom

$$x \circ (y \circ z) = z \circ \big(y \circ (z \circ x)\big)$$

and containing at least 4 elements are A_2A^3-algebras and

(2) $A^{5'}$-algebras fulfilling the axiom

$$x \circ (y \circ z) = z \circ \Big(y \circ \big(x \circ (x \circ z)\big)\Big)$$

and containing at least 6 elements are $A_2A^{5'}$-algebras (see [2]).

Problem 1. Is it true that each A^k-algebra fulfilling the axiom

$$x \circ (y \circ z) = z \circ \big(y \circ (z^{k-3+\varphi(1)}[a])\big)$$

and containing at least $k+1$ elements is an A_2A^k-algebra? **(P 597)**

It is seen from (1), (2) and Theorem 2 of § 1 that the answer is in the affirmative for all A^k-algebras for which $k < 7$ or for which $\varphi(1) = 2$.

The author does not know the answer to the following question:

PROBLEM 2. Is it true that each subset of an A_2A^k-algebra $\mathfrak{A}$ containing m elements which is not generated by any proper subset generates a subset of A containing exactly k^{m-1} elements? (**P 598**)

REFERENCES

[1] L. Szamkołowicz, *Remarks on finite regular planes*, Colloquium Mathematicum 10 (1963), p. 31-37.

[2] — *Sulla generalizzazione del concetto delle algebre* A_n^3, Accademia Nazionale dei Lincei, Rendiconti, S. VIII, 38 (1965), p. 810-814.

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