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ON CERTAIN SINGULAR MEASURES

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0. Hewitt and Kakutani [2] discovered a large class of probability measures λ on a locally compact abelian group G such that

(1) $\lambda(E) = \lambda(-E)$ for Borel subsets E.

(2) For integers $m > n \geqslant 1$, every translate $b+S(\lambda^n)$, of the closed support of λ^n, has λ^m measure 0.

The concept of independence is basic in [2], but measures λ are constructed in [4] with no algebraic tools. Here we present a third method quite different from these, wherein G is always the real line. We define

$$\|t\| \equiv \inf |t-n|, \qquad n = 0, \pm 1, \ldots$$

1. Let $\theta > 1$ be a real number with the property

$$\|\theta^k\| \leqslant Ar^k, \qquad 0 < r < 1,\ 1 \leqslant k. \tag{3}$$

The numbers θ are treated at length by Salem [5]. Let

$$T: \quad 0 < p_1 < p_2 < \ldots < p_k < p_{k+1} < p_{k+2} < \ldots$$

be a sequence of integers with many "isolated" members:

$$\sup_k \inf(p_{k+1} - p_k, p_{k+2} - p_{k+1}) = \infty. \tag{4}$$

Finally, λ is the measure whose Fourier-Stieltjes transform is

$$\hat{\lambda}(u) \equiv \int e^{-iut}\lambda(dt) = \prod_{k=1}^{\infty} \cos(u\theta^{-p_k}), \qquad -\infty < u < \infty.$$

THEOREM 1. *λ has properties* (1) *and* (2).

From a deep method of Erdös we have a complementary result.

THEOREM 2. *There exists a sequence T with property* (4) *and a number $d > 1$ such that for almost all $\varphi \in (1, d)$ the measure μ with transform*

$$\hat{\mu}(u) = \prod_{k=1}^{\infty} \cos(u\varphi^{-p_k})$$

has a continuous density. Thus μ^4 and μ^2 are not mutually singular.

Returning to the first result, we say that a *measure ν has property* (S) (for Šreĭder), if for each complex number z of modulus $\leqslant 1$, there is a sequence of integers u_n such that

$$\lim_{n\to\infty} \int_F e^{-2\pi i u_n t} \nu(dt) = z\nu(F)$$

for every Borel set F (see [3]).

THEOREM 3. *For almost every real number y, the y-translate of λ, $\lambda_y(E) \equiv \lambda(E-y)$, has property* (S).

2. Proof of Theorem 1. Property (1) is obvious for λ and it can be used to eliminate consideration of the element b in (2). For the symmetry of λ yields

$$\lambda^{2m}\big(S(\lambda^{2n})\big) = \lambda^{2m}\big(S(\lambda^n)-S(\lambda^n)\big) \geqslant \big(\lambda^m\big(S(\lambda^n)+b\big)\big)^2.$$

It is convenient now to write λ^{2m} in a more analytical form. Let A be the space of sequences $X = (X_1, X_2, \ldots, X_k, \ldots)$, each X_k assuming integral values in $[-2m, 2m]$. A measure P is defined in A by requiring that the X_k's are mutually independent and

$$P\{X_k = 2s\} = 4^{-m}\binom{2m}{m+s}, \quad s = -m, -m+1, \ldots, m.$$

Then λ^{2m} is the distribution of the random variable

$$Z = \sum_{k=1}^{\infty} \theta^{-p_k} X_k.$$

We can determine infinitely many integers q, and integers $j = j(q)$ attached to them, so that

$$X_{q+1} \neq X'_{q+1} \Rightarrow \theta^j Z - \theta^j Z' \not\equiv 0 \ (\mathrm{mod}\ 1).$$

Suppose in fact that

$$\frac{1}{10m\theta} \leqslant \theta^{j-p_{q+1}} \leqslant \frac{1}{10m} \tag{5}$$

and

$$(6) \qquad \|\theta^j Z - \theta^{j-p_{q+1}} X_{q+1}\| < \frac{1}{20m\theta}$$

for all X. Then, if $X_{q+1} \neq X'_{q+1}$,

$$\frac{1}{10m\theta} \leqslant \theta^{j-p_{q+1}} |X_{q+1} - X'_{q+1}| \leqslant \frac{4m}{10m},$$

$$0 < |\theta^j Z - \theta^j Z' - n| < \frac{2}{5} - \frac{1}{10m\,\theta} < \frac{1}{2}$$

for some integer n. The exponent j can always be chosen so as to satisfy (5). Now

$$\begin{aligned} \|\theta^j Z - \theta^{j-p_{q+1}} X_{q+1}\| &\leqslant 2m \sum_{k\leqslant q} \|\theta^{j-p_k}\| + 2m \sum_{q+1<k} \theta^{j-p_k} \\ &\leqslant 2mA \sum_{k\leqslant q} r^{j-p_k} + 2m \sum_{k=p_{q+2}-j}^{+\infty} \theta^{-k} \\ &\leqslant 2mA \sum_{k=j-p_q}^{+\infty} r^k + 2m \sum_{k=p_{q+2}-j}^{+\infty} \theta^{-k}. \end{aligned}$$

By (5), $j - p_{q+1}$ is bounded for all q, and using (4), both $j - p_q$ and $p_{q+2} - j$ can be made arbitrarily large and positive. Then (6) is attained. To prove Theorem 1 observe that if $X_{q+1} > 2n$, then $Z \notin S(\lambda^{2n})$; hence $P\{Z \in S(\lambda^{2n})\} = 0$.

Proof of Theorem 2. Here is a statement of the result of Erdös [1]. For numbers $u > 0$ and $\varphi > 1$ let $N(u, \varphi)$ be the number of integers k for which

$$k \geqslant 1, \qquad |\cos(u\varphi^{-k})| < \cos\frac{1}{30}.$$

For any $h \geqslant 1$ there is a $c = c(h) > 1$ such that

$$e^{-N(u,\varphi)} = 0(u^{-h}), \qquad u \to +\infty,$$

for almost all $\varphi \in (1, c)$.

The sequence T depends on a random sequence

$$\xi = (\xi_1, \xi_2, \ldots, \xi_k, \ldots)$$

of 0's and 1's distributed as in the coin-tossing game. Formally, $T = T(\xi) = \{k\colon \xi_k = 1\}$, and property (4) is almost certain. The transform μ now takes the form

$$\hat{\mu}(u) = \prod_{k=1}^{\infty} \cos(u\xi_k\varphi^{-k}).$$

If φ is fixed, $\hat{\mu}(u)$ is regarded as a function of ξ, and E denotes expectation

$$E(|\hat{\mu}(u)|) = \prod_{k=1}^{\infty} \tfrac{1}{2}(1 + |\cos(u\varphi^{-k})|).$$

Using Erdös' theorem, there is some $d > 1$ such that for almost all $\varphi \in (1, d)$

$$\int_{-\infty}^{\infty} |u|^3 E(|\hat{\mu}(u)|)\, du < \infty,$$

hence for almost all ξ

$$\int_{-\infty}^{\infty} |u|^3 |\hat{\mu}(u)|\, du < \infty.$$

The statement of the theorem now follows by applying Fubini's Theorem to the variables ξ and φ.

Proof of Theorem 3. We use again the space A of the first proof but set

$$P\{X_k = 1\} = P\{X_k = -1\} = \tfrac{1}{2}.$$

The distribution of Z is now λ. Let n_1 be a positive integer; by the argument of the first theorem there is a sequence of integers j, converging to infinity such that

$$\lim_{j\to\infty} \|\theta^{p_j - n_1} Z - \theta^{-n_1} X_j\| = 0$$

uniformly in A. This holds true if $\theta^{p_j - n_1}$ is replaced by the nearest integer, say u_j, and if the numbers in the braces are multiplied by a fixed integer n_2. Hence

$$\lim_{u_j\to\infty} |\exp - (2\pi i n_2 u_j Z) - \exp(-2\pi i n_2 \theta^{-n_1} X_j)| = 0.$$

Let $f(X)$ be a complex function of only finitely many of the coordinates X_k. Then

$$\lim_{u_j\to\infty} E\big(f(X) \exp - (2\pi i n_2 u_j Z)\big) = \cos 2\pi n_2 \theta^{-n_1} E(f).$$

By an easy continuity argument this is valid for any integrable function $f(X)$, in particular, the indicator function of the set $\{X \in F\}$, F Borel in $(-\infty, \infty)$. The last equation is then

$$\lim_{u_j\to\infty} \int_F e^{-2\pi i n_2 u_j t} \lambda(dt) = \cos 2\pi n_2 \theta^{-n_1} \lambda(F).$$

When λ is replaced by λ_y, the left-hand side is multiplied by $e^{-2\pi i n_2 u_j y}$, and the right is changed to $\lambda_y(F)$.

The numbers $n_2\theta^{-n_1}(n_1, n_2 \geqslant 1)$ are dense (mod 1), so that λ_y has property (S) provided that for each sequence $\{n_2 u_j\}_{j=1}^{\infty}$ constructed above, $\{n_2 u_j y\}_{j=1}^{\infty}$ is dense (mod 1). The sequences $\{n_2 u_j\}$ are countable in number so it is enough to prove the density condition for almost every y and each one separately. If then $\{n_2 u_j y\}$ is not dense (mod 1), there is a pair of rational numbers r_1, r_2 $(0 < r_1 < r_2 < 1)$ such that

$$n_2 u_j y \notin (r_1, r_2) (\mathrm{mod}\, 1) \quad \text{for} \quad j \geqslant 1.$$

The last line defines an "H-set" (Rajchman) and this has measure 0 (see Zygmund [6], p. 268). The proof is complete.

REFERENCES

[1] P. Erdös, *On the smoothness properties of a family of Bernoulli convolutions*, American Journal of Mathematics 62 (1940), p. 180-186.

[2] E. Hewitt and S. Kakutani, *A class of multiplicative linear functionals on the measure algebra of a locally compact Abelian group*, Illinois Journal of Mathematics 4 (1960), p. 553-574.

[3] — *Some multiplicative linear functionals on $M(G)$*, Annals of Mathematics (2) 79 (1964), p. 489-505.

[4] R. Kaufman, *Some measures determined by mappings of the Cantor set*, Colloquium Mathematicum 19 (1968), p. 77-83.

[5] R. Salem, *Algebraic numbers and Fourier analysis*, Boston 1963.

[6] A. Zygmund, *Trigonometrical series*, Warsaw 1935.

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