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A REMARK ON CONTINUITY OF ONE-PARAMETER SEMI-GROUPS OF OPERATORS

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1. Let X be a Banach space and $\mathscr{M}$ — the linear space of all X-valued Bochner measurable functions defined everywhere on $(0, \infty)$. Denote by $\mathscr{L}(X; X)$ the family of all linear continuous operators of X into X and by $\mathscr{L}(X; \mathscr{M})$ the family of all operators of X into $\mathscr{M}$ which become linear after identification of almost everywhere equal functions in $\mathscr{M}$. If $S \in \mathscr{L}(X; \mathscr{M})$ and $x \in X$, then Sx is a function belonging to $\mathscr{M}$, defined everywhere on $(0, \infty)$. For any $t \in (0, \infty)$ we denote by $S_t x$ the value of Sx at t.

Let $S \in \mathscr{L}(X; \mathscr{M})$ be such that

(i) for every $x \in X$ the equality $S_t(S_s x) = S_{t+s} x$ holds for almost every pair $(t, s) \in (0, \infty)^2$;

(ii) there is a function h defined on $(0, \infty)$, with values in $[0, \infty)$, such that for every $x \in X$ the inequality $\|S_t x\| \leqslant h(t)\|x\|$ holds for almost every $t \in (0, \infty)$.

If for every $x \in X$, the equality in (i) and the inequality in (ii) hold for every $(t, s) \in (0, \infty)^2$ and every $t \in (0, \infty)$, respectively, and if S is a linear operator of X into $\mathscr{M}$, then S is simply a strongly measurable semi-group defined on $(0, \infty)$ of bounded linear operators of X into X, and by a theorem of Phillips [3], this semi-group is strongly continuous. But, as pointed out by Feller [1], it is more natural to assume that (i) and (ii) hold only almost everywhere. The following theorem goes in this direction:

THEOREM 1. *If $S \in L(X; M)$ satisfies* (i) *and* (ii), *then there exists a strongly continuous semi-group $\{T_t: t > 0\} \subset \mathscr{L}(X; X)$ such that for every $x \in X$ the equality $S_t x = T_t x$ holds for almost every $t \in (0, \infty)$.*

In this paper m will denote the Lebesgue measure, m_* — the inner Lebesgue measure and m^* — the outer one.

2. The result of Phillips may be deduced from Theorem 1 as follows. If $\{S_t: t > 0\} \subset \mathscr{L}(X; X)$ is a strongly measurable semigroup, then by

Theorem 1 there is a strongly continuous semigroup $\{T_t: t > 0\} \subset \mathscr{L}(X; X)$ such that, for every $x \in X$,

$$(*) \qquad \mathrm{m}\big((0, \infty)\setminus E_x\big) = 0,$$

where $E_x = \{t: t \in (0, \infty), S_t x = T_t x\}$.

For every $x \in X$ and $s > 0$ we have

$$\bigcap_{r>0} E_{T_r x} \subset E_{T_s x},$$

where r is rational.

Indeed, let r_n, $n = 1, 2, \ldots$, be a sequence of positive rationals converging to $s > 0$ and let

$$t \in \bigcap_{n=1}^{\infty} E_{T_{r_n} x};$$

then $T_{r_n} x$ converges to $T_s x$ and $S_t T_{r_n} x = T_t T_{r_n} x$ for every $n = 1, 2, \ldots$ so that

$$S_t T_s x = \lim_{n\to\infty} S_t T_{r_n} x = \lim_{n\to\infty} T_t T_{r_n} x = T_t T_s x \quad \text{and} \quad t \in E_{T_s x}.$$

Hence, if for any $x \in X$ we put

$$F_x = \bigcap_{s>0} E_{T_s x},$$

then $F_x = \bigcap_{r>0} E_{T_r x}$ (r rational), so that, by $(*)$,

$$(**) \qquad \mathrm{m}\big((0, \infty)\setminus F_x\big) = 0.$$

It follows from $(*)$ and $(**)$ that, for any $x \in X$, every $t > 0$ may be written as a sum $t = \sigma + \tau$, where $\sigma \in E_x$ and $\tau \in F_x$. For, if this is not true for some $t > 0$, then $(0, t) \subset (0, t)\setminus E_x \cup (0, t)\setminus(t - F_x)$, so that

$$t \leqslant \mathrm{m}\big((0, t)\setminus E_x\big) + \mathrm{m}\big((0, t)\setminus(t - F_x)\big) = \mathrm{m}\big((0, t)\setminus E_x\big) + \mathrm{m}\big((0, t)\setminus F_x\big) = 0.$$

But, if $\sigma \in E_x$ and $\tau \in F_x$, then

$$S_{\sigma+\tau} x = S_\tau S_\sigma x = S_\tau T_\sigma x = T_\tau T_\sigma x = T_{\sigma+\tau} x,$$

so that $\sigma + \tau \in E_x$. Hence $E_x = (0, \infty)$ for every $x \in E$. The semigroups $\{S_t\}$ and $\{T_t\}$ are therefore identical and the theorem of Phillips is proved.

3. For the proof of Theorem 1 we need some lemmas. Recall that a point $p \in R^n$ is called a *density point* of a set $E \subset R^n$ if

$$\lim_{I\to 0} (\mathrm{m} I)^{-1} \mathrm{m}_*(E \cap I) = 1,$$

where I runs over n-cubes containing p.

An extended real-valued function f defined on a set $E \subset R^n$ is called *approximatively lower semi-continuous at point* $p \in E$ if, for every $a \in [-\infty, \infty]$, the inequality $a < f(p)$ implies that p is a density point of the set $\{q: q \in E, f(q) > a\}$. We say that f is *approximatively upper semi-continuous at* p, if $(-f)$ is approximatively lower semi-continuous at p. It is known that a set $E \in R^n$ is measurable if and only if almost every its point is its density point (see [2]). This implies that an extended real-valued function defined on a measurable set $E \subset R^n$ is measurable if and only if it is approximatively lower semi-continuous at almost every $p \in E$.

LEMMA 1. *Let F be a family of extended real-valued measurable functions defined on a measurable set $E \subset R^n$. Assuming that* sup $\emptyset = -\infty$, *for any $p \in E$ put*

$$M_F(p) = \sup\{f(p): f \in F, f \text{ is approximatively lower semi-continuous at } p\}.$$

Then

(a) *the function M_F is measurable,*

(b) *for every $f \in F$ the inequality $f(p) \leqslant M_F(p)$ holds almost everywhere on E,*

(c) *if h is an extended real-valued function defined on E such that for every $f \in F$ we have $f(p) \leqslant h(p)$ almost everywhere on E, then $M_F(p) \leqslant h(p)$ almost everywhere on E.*

Proof. As it is easy to see, M_F is approximatively lower semi-continuous at every density point of E, and hence it is measurable. Property (b) follows from the fact that any $f \in F$ is approximatively lower semi-continuous at almost every $p \in E$. To prove (c), suppose that $M_F(p) \leqslant h(p)$ does not hold almost everywhere on E. Then there is an $\varepsilon > 0$ such that the set

$$A = \left\{p: p \in E, \min\left(M_F(p), \frac{1}{\varepsilon}\right) - \varepsilon > h(p)\right\}$$

has positive outer measure. Put

$$C = \{p: p \in E, M_F \text{ is approximatively upper semi-continuous at } p\}.$$

$$D = \{p: p \in A, \lim_{I \to 0} (\mathrm{m}I)^{-1}\mathrm{m}^*(A \cap I) = 1\},$$

where, in the definition of D, I runs over n-cubes containing p.

Then $m(E \setminus C) = 0$ by measurability of M_F and $\mathrm{m}(A \setminus D) = 0$ by the Lebesgue density theorem. Since $A \subset (C \cap D) \cup (A \setminus D) \cup (E \setminus C)$, this implies that $\mathrm{m}^*(C \cap D) = \mathrm{m}^* A > 0$ and so $C \cap D \neq 0$.

Let $p_0 \epsilon C \cap D$. Then $p_0 \epsilon A$, so that $M_F(p_0) > -\infty$ and, therefore, there exists a function $f \epsilon F$, which is approximatively lower semi-continuous at p_0 and satisfies the inequality $f(p_0) > \min\big(M_F(p_0), 1/\varepsilon\big) - \varepsilon$.

Then p_0 is a density point of the set

$$B = \left\{p: p \epsilon E, f(p) > \min\left(M_F(p), \frac{1}{\varepsilon}\right) - \varepsilon\right\}.$$

Since B is measurable, we have $m^*(A \cap B \cap I) = m^*(A \cap I) - m^*\big((A \cap I) \setminus B\big) \geqslant m^*(A \cap I) - m(I \setminus B)$ for any n-cube I containing p_0. Because $p_0 \epsilon D$ and it is a density point of B, we infer from this, by taking I sufficiently small, that $m^*(A \cap B) > 0$. But $f(p) > h(p)$ for $p \epsilon A \cap B$, so that $f(p) \leqslant h(p)$ does not hold almost everywhere on E. Hence (c) is proved.

LEMMA 2 (1). *Let ω be an extended real-valued function measurable on $(0, \infty)$ such that $\omega(t) < \infty$ for almost every $t \epsilon (0, \infty)$ and*

$$\omega(t+s) \leqslant \omega(t) + \omega(s)$$

for almost every pair $(t,s) \epsilon (0, \infty)^2$. Then ω is essentially bounded from above on every interval $[a, b]$ such that $0 < a < b < \infty$.

Proof. Put $\Delta = \{(u, v): 0 < v < u < \infty\}$. Using the mapping $(t, s) \to (u, v) = (t+s, s)$, we see that $\omega(u) \leqslant \omega(u-v) + \omega(v)$ for almost every pair $(u, v) \epsilon \Delta$. Hence, if we put

$$E_u = \{v: 0 < v < u, \omega(u) > \omega(u-v) + \omega(v)\}$$

for any $u \epsilon (0, \infty)$ and

$$Z = \{u: u \epsilon (0, \infty), mE_u > 0\},$$

then $mZ = 0$. Let $0 < a < b < \infty$. We say that

$$(*) \qquad \sup\{\omega(u): u \epsilon [a, b] \setminus Z\} < \infty.$$

Indeed, if no, then there is a sequence $u_n, n = 1, 2, \ldots$, such that $u_n \epsilon [a, b] \setminus Z$ and $\omega(u_n) \geqslant 2n$. For any $n = 1, 2, \ldots$ put

$$F_n = \{v: v \epsilon (0, b), \omega(v) \geqslant n\}.$$

If $0 < v < u_n$ and $v \notin E_{u_n}$, then $\omega(v) + \omega(u_n - v) \geqslant \omega(u_n) \geqslant 2n$, so that $v \epsilon F_n$ or $u_n - v \epsilon F_n$. Hence $(0, u_n) \setminus E_{u_n} \subset F_n \cup (u_n - F_n)$, which, because of $u_n \notin Z$ and in consequence, of $mE_{u_n} = 0$ implies that

$$a \leqslant u_n = m\big((0, u_n) \setminus E_{u_n}\big) \leqslant mF_n + m(u_n - F_n) = 2mF_n.$$

(1) This lemma is an adaptation for our purposes of a known lemma used also by Phillips [3].

Since $\mathrm{m}F_n \leqslant b$, $F_1 \supset F_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} F_n = F_\infty$, it follows that

$$\mathrm{m}F_\infty = \inf_{n=1,2,\ldots} \mathrm{m}F_n \geqslant \frac{1}{2}a > 0.$$

But this is impossible. Hence (*) must hold and the Lemma is proved.

4. Proof of Theorem 1. Let $S \epsilon \mathscr{L}(X; \mathscr{M})$ satisfy (i) and (ii). Assuming that $\sup \emptyset = -\infty$, put, for any $t > 0$,

$$k(t) = \sup\{\|S_t x\| : x \epsilon X,\ \|x\| \leqslant 1, \text{ the function } \tau \to \|S_\tau x\| \text{ is approximatively lower semi-continuous at } \tau = t\}.$$

Then, as it follows from Lemma 1, k is a measurable function on $(0, \infty)$, $k(t) \leqslant h(t) < \infty$ for almost every $t \epsilon (0, \infty)$, and

(ii)′ for every $x \epsilon X$ the inequality $\|S_t x\| \leqslant k(t)\|x\|$ holds for almost every $t \epsilon (0, \infty)$.

If $x \epsilon X$ is arbitrarily fixed, then, by (i) and (ii)′, for almost every fixed $t \epsilon (0, \infty)$ we have

$$\|S_t x\| \leqslant k(t)\|x\|$$

and

$$\|S_{t+s} x\| = \|S_s S_t x\| \leqslant k(s)\|S_t x\| \leqslant k(s) k(t)\|x\|$$

for almost every $s \epsilon (0, \infty)$. Because the first and the last members of the former inequality are two-dimensionally measurable, hence, for every $x \epsilon X$, we have

$$\|S_{t+s} x\| \leqslant k(t) k(s)\|x\|$$

for almost every pair $(t, s) \epsilon (0, \infty)^2$.

For any pair $(t, s) \epsilon (0, \infty)^2$ put

$$n(t, s) = \sup\{\|S_{t+s} x\| : x \epsilon X,\ \|x\| \leqslant 1, \text{ the function } (\tau, \sigma) \to \|S_{\tau+\sigma} x\| \text{ is approximatively lower semicontinuous at } (\tau, \sigma) = (t, s)\}.$$

Then, by the preceding inequality and by (c) of Lemma 1, we have

$$n(t, s) \leqslant k(t) k(s)$$

for almost every pair $(t, s) \epsilon (0, \infty)^2$. On the other hand, for any $x \epsilon X$, the function $(\tau, \sigma) \to \|S_{\tau+\sigma} x\|$ is two-dimensionally approximatively lower semi-continuous at $(\sigma, \tau) = (t, s)$, iff the function $\tau \to \|S_\tau x\|$ is one-dimensionally approximatively lower semi-continuous at $\tau = t + s$, so that

$$n(t, s) = k(t + s)$$

for every $(t, s) \in (0, \infty)^2$. Hence $k(t+s) \leqslant k(t)k(s)$ for almost every pair $(t, s) \in (0, \infty)^2$, which, by applying Lemma 2 to $\omega = \log k$, implies that

$$\operatorname*{ess\,sup}_{[a,b]} k(t) < \infty$$

for every a and b satisfying $0 < a < b < \infty$.

Let $x \in X$ be arbitrarily fixed. Then for almost every $u \in (0, \infty)$ fixed and almost every $v \in (0, \infty)$ fixed we have

$$\|S_{\tau+u}x - S_{\tau+v}x\| = \|S_\tau(S_u x - S_v x)\| \leqslant k(\tau)\|S_u x - S_v x\|$$

for almost every $\tau \in (0, \infty)$. So, by the three-dimensional measurability of the first and the last member, we have

$$\|S_{\tau+u}x - S_{\tau+v}x\| \leqslant k(\tau)\|S_u x - S_v x\|$$

for almost every triple $(\tau, u, v) \in (0, \infty)^3$. From this, using the mapping $(\tau, u, v) \to (\tau, t, s) = (\tau, u+\tau, v+\tau)$, we obtain that the inequality

$$\|S_t x - S_s x\| \leqslant k(\tau)\|S_{t-\tau}x - S_{s-\tau}x\| \tag{4.1}$$

holds three-dimensionally almost everywhere on the set

$$\{(\tau, t, s): (t, s) \in (0, \infty)^2, 0 < \tau < \min(t, s)\}.$$

Hence, if for any $x \in X$ and $t > 0$ we put

$$E_{x,t} = \{s: s \in (0, \infty), \text{ (4.1) holds for almost every } \tau \in (0, \min(t,s))\}$$

and

$$E_x = \{t: t \in (0, \infty), \mathrm{m}((0, \infty)\setminus E_{x,t}) = 0\},$$

then

$$\mathrm{m}((0, \infty)\setminus E_x) = 0 \quad \text{for every } x \in X. \tag{4.2}$$

For any $x \in X$, $0 < a < b < \infty$ and $\delta > 0$ put

$$\omega_{x;a,b}(\delta) = \frac{3}{a} \operatorname*{ess\,sup}_{[a/3,2a/3]} k(t) \sup \int_{a/3}^{2a/3} \|S_{t-\tau}x - S_{s-\tau}x\| d\tau,$$

where sup is taken over $s, t \in [a, b]$ such that $|s-t| \leqslant \delta$. Then, since

$$\operatorname*{ess\,sup}_{[a/3,b-a/3]} k(t) < \infty,$$

the function $\sigma \to S_\sigma x$ is Bochner integrable on $[\frac{1}{3}a, b-\frac{1}{3}a]$ and so $t \to \{\tau \to S_{t-\tau}x\}$ is a continuous mapping of $[a, b]$ into the space of X-valued Bochner integrable functions on $[\frac{1}{3}a, \frac{2}{3}a]$, normed by

$$\|x(\cdot)\| = \int_{a/3}^{2a/3} \|x(\tau)\|_X d\tau.$$

Hence

$$\lim_{\delta\to+0}\omega_{x;a,b}(\delta) = 0 \tag{4.3}$$

for every $x\in X$ and $0 < a < b < \infty$.

If $x\in X$, $0 < a < b < \infty$ and $t_1, t_2\in[a, b]\cap E_x$, then

$$m([a, b]\cap E_{x,t_1}\cap E_{x,t_2}) = b-a$$

and so, for any $\varepsilon > 0$ there is an $s\in[a, b]\cap E_{x,t_1}\cap E_{x,t_2}$ such that $\omega_{x;a,b}(|t_1-s|) < \varepsilon$ and $|s-t_2| \leqslant |t_1-t_2|$. By the definition of $E_{x,t}$, for $i = 1, 2$, we have

$$\|S_{t_i}x-S_sx\| \leqslant k(\tau)\|S_{t_i-\tau}x-S_{s-\tau}x\|$$

for almost every $\tau\in[\frac{1}{3}a, \frac{2}{3}a]$, so that, integrating both parts with respect to τ on $[\frac{1}{3}a, \frac{2}{3}a]$, we obtain

$$\|S_{t_i}x-S_sx\| \leqslant \omega_{x;a,b}(|t_i-s|)$$

for $i = 1, 2$. So

$$\|S_{t_1}x-S_{t_2}x\| \leqslant \omega_{x;a,b}(|t_2-s|)+\varepsilon \leqslant \omega_{x;a,b}(|t_1-t_2|)+\varepsilon$$

and, $\varepsilon > 0$ being arbitrary, we see that

$$\|S_{t_1}x-S_{t_2}x\| \leqslant \omega_{x;a,b}(|t_1-t_2|) \tag{4.4}$$

for any $x\in X$, $0 < a < b < \infty$ and $t_1, t_2\in[a, b]\cap E_x$.

Now we can easily define the semi-group $\{T_t: t > 0\}\subset \mathscr{L}(X, X)$ satisfying the statement of Theorem 1. Indeed, for any $x\in X$ there is a unique X-valued function Tx strongly continuous on $(0, \infty)$ such that $Tx(t) = S_tx$ for every $t\in E_x$. Consequently, for any $x, y\in X$ and $\alpha, \beta\in R^1$, we have

$$T(\alpha x+\beta y)(t) = S_t(\alpha x+\beta y) = \alpha S_tx+\beta S_ty = \alpha Tx(t)+\beta Ty(t)$$

for almost every $t\in(0, \infty)$ and so, by continuity,

$$T(\alpha x+\beta y)(t) = \alpha Tx(t)+\beta Ty(t)$$

for every $t\in(0, \infty)$. Hence the equality

$$T_tx = Tx(t)$$

for $x\in X$ and $t\in(0, \infty)$ defines a family $\{T_t: t > 0\}$ of linear operators of X into X. By (ii)′, for any $x\in X$ such that $\|x\| \leqslant 1$ and any $0 < a < b < \infty$ we have

$$\|T_tx\| \leqslant \operatorname*{ess\,sup}_{[a,b]} k(t) < \infty$$

for almost every $t \in [a, b]$ and so, by continuity, this inequality holds for every $t \in [a, b]$. This implies that $\{T_t: t > 0\} \subset \mathscr{L}(X; X)$. At last, if $x \in X$ is arbitrarily fixed, then for almost every fixed $t \in (0, \infty)$ we have $T_t x = S_t x$, $T_s S_t x = S_s S_t x = S_{s+t} x$ for almost every $s \in (0, \infty)$, and $S_{s+t} x = T_{s+t} x$ for almost every $s \in (0, \infty)$, so that $T_s T_t x = T_{t+s} x$ for almost every $s \in (0, \infty)$. It follows by continuity that, for every $x \in X$, $T_s T_t x = T_{s+t} x$ for every pair $(t, s) \in (0, \infty)^2$. Hence $\{T_t: t > 0\}$ is a semi-group and the proof is completed.

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