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SOME PRESERVATION THEOREMS

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In this paper we prove some conjectures (Theorem 1.2 and Corollaries 2.4 and 2.5) raised by Jan Mycielski and C. Ryll-Nardzewski in 1961 which were already proved about that time by D. S. Scott but never published. A typical result (Theorem 1.6) says that an elementary sentence is preserved under endomorphisms if and only if it is logically equivalent to a conjunction of disjunctions of positive sentences and universal sentences. We shall use the general method of H. J. Keisler (see [2]) which is by now a standard tool for treating such problems. We do not know the original proofs of D. S. Scott. We only know that in his proof of 1.2 he used a certain generalization of the Craig-Lyndon interpolation theorem; our proof is different.

1. Operations on structures. First we prove some theorems concerning particular situations which will be generalized later in Theorem 1.8.

Let $\mathfrak{A} = \langle A, R_{ij}^{\mathfrak{A}}\rangle_{i<n, j<m}$ and $\mathfrak{B} = \langle B, R_{ij}^{\mathfrak{B}}\rangle_{i<n, j<m}$ be two similar relational structures. An n-tuple of functions $\boldsymbol{h} = \langle h_0, \ldots, h_{n-1}\rangle$ is called a *sliced homomorphisms* of $\mathfrak{A}$ onto $\mathfrak{B}$, if $h_i(A) = B$ for each $i<n$ and $\langle h_i(a_0), \ldots, h_i(a_{r(i,j)-1})\rangle \in R_{ij}^{\mathfrak{B}}$ for $\langle a_0, \ldots, a_{r(i,j)-1}\rangle \in R_{ij}^{\mathfrak{A}}$, ($r(i,j)$ denotes the number of places of $R_{ij}^{\mathfrak{A}}$).

If h_i is one-to-one and $\boldsymbol{h}^{-1} = \langle h_0^{-1}, \ldots, h_{n-1}^{-1}\rangle$ is also a sliced homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$, then $\boldsymbol{h}$ is called a *sliced isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$ (1).

The terminology and notation in this section follows [6]. Let us denote by $\mathscr{L}_i = \langle \boldsymbol{P}_{ij}, v_{in}\rangle_{n<\omega, j<m}$ a first order language, where $\boldsymbol{P}_{ij}$ are interpreted as $R_{ij}^{\mathfrak{A}}$ and v_{in} are running over A.

LEMMA 1.1 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be elementarily $\mathfrak{m}$-compact structures* (2) *of power at most $\mathfrak{m}^+$. A neccesary and sufficient condition for $\mathfrak{B}$ to be a sliced isomorphic image of $\mathfrak{A}$ is that:*

$$Th_i(\mathfrak{A}) = Th_i(\mathfrak{B}) \quad \textit{for} \quad i = 0, 1, \ldots, n-1,$$

(1) This notion is due to Jan Mycielski.

(2) We recall that for languages having at most $\mathfrak{m}$ non-logical constants $\mathfrak{A}$ is elementarily $\mathfrak{m}$-compact (see [6]) if and only if it is $\mathfrak{m}^+$-saturated (cf. [1] and [4]).

where $Th_i(\mathfrak{X})$ denotes the complete theory of the structure $\mathfrak{X}$ expressed in the language $\mathcal{L}_i$.

Proof. The necessity is obvious.

Let us consider the structures $\mathfrak{A}_i = \langle A, R_{ij}^{\mathfrak{A}}\rangle_{j<m}$ and $\mathfrak{B}_i = \langle B, R_{ij}^{\mathfrak{B}}\rangle_{j<m}$ that are also elementarily $\mathfrak{m}$-compact of power at most $\mathfrak{m}^+$. Since $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are elementarily equivalent, thus, using a theorem of Morley and Vaught [3], there is an isomorphism h_i of $\mathfrak{A}_i$ onto $\mathfrak{B}_i$. Thus $\boldsymbol{h} = \langle h_0, \ldots, h_{n-1}\rangle$ is the required sliced isomorphism.

Now, we can obtain the following theorem conjectured by Jan Mycielski and proved by D. S. Scott:

THEOREM 1.2. *Let ϑ be a sentence. Then the following conditions are equivalent:*

(i) *If ϑ holds in a structure $\mathfrak{C}$, then ϑ holds in every sliced isomorph of $\mathfrak{C}$.*

(ii) *ϑ is logically equivalent to a conjunction of disjunctions of sentences each of which is expressed in one of the languages $\mathcal{L}_i$ $(i = 0, 1, \ldots, n-1)$.*

Proof. It is easy to see that (ii) implies (i). Now we assume (i) and for the moment, that $2^{\mathfrak{m}} = \mathfrak{m}^+$. Let $\mathfrak{C}$ and $\mathfrak{D}$ be structures such that $Th(C) \cap \Sigma \subseteq Th(D)$, where Σ is the set of all sentences described in (ii), ϑ is a sentence satisfying (i) and $\mathfrak{C} \models \vartheta$. By Theorem 1 of [5] there are elementarily $\mathfrak{m}$-compact structures $\mathfrak{A}$ and $\mathfrak{B}$, both of power $\mathfrak{m}^+$, which are elementary extensions of $\mathfrak{C}$ and $\mathfrak{D}$, respectively. Thus $Th(\mathfrak{A}) \cap \Sigma \subseteq Th(\mathfrak{B})$ and we have $Th_i(\mathfrak{A}) = Th_i(\mathfrak{B})$. By Lemma 1.1, $\mathfrak{B}$ is an image of $\mathfrak{A}$ by a sliced isomorphism. Since $\mathfrak{A} \models \vartheta$, so, by *(i)*, $\mathfrak{B} \models \vartheta$ and thus $\mathfrak{D} \models \vartheta$. By Lemma 1.1 of [2], we conclude that $\vartheta \in \Sigma$ and (ii) follows.

To eliminate the Generalized Continuum Hypothesis from our argument we remark first that property (i) can be expressed in a purely syntactical form. Now we can eliminate the assumption $2^{\mathfrak{m}} = \mathfrak{m}^+$ by using arithmetization and the result of Gödel saying that any arithmetical statement about natural numbers which can be proved with the aid of the Generalized Continuum Hypothesis (or even the Axiom of Constructibility) can be proved without it.

LEMMA 1.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be elementarily $\mathfrak{m}$-compact structures of power at most $\mathfrak{m}^+$. A neccesary and sufficient condition for $\mathfrak{B}$ to be a sliced homomorphic image of $\mathfrak{A}$ is that*

$$Th_i(\mathfrak{A}) \cap \Pi \subseteq Th_i(\mathfrak{B}) \quad \textit{for} \quad i = 0, 1, \ldots, n-1,$$

where Π is the set of all sentences logically equivalent to positive sentences.

We omit the proof since it is quite similar to the proof of Lemma 1.1 if we use a theorem of Keisler (see [2], Theorem 3.1) on homomorphisms of elementarily $\mathfrak{m}$-compact structures.

THEOREM 1.4. *Let ϑ be a sentence. Then the following conditions are equivalent*:

(i) *If ϑ holds in a structure $\mathfrak{C}$, then ϑ holds in every sliced homomorphic image of $\mathfrak{C}$.*

(ii) *ϑ is logically equivalent to a conjunction of disjunctions of positive sentences each of which is expressed in one of the languages $\mathscr{L}_i$ $(i = 0, 1, \ldots, \ldots, n-1)$,* or $\vdash \neg \vartheta$.

The proof of the above Theorem is analogous to the proof of Theorem 1.2.

LEMMA 1.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be elementarily $\mathfrak{m}$-compact structures of power at most $\mathfrak{m}^+$. A necessary and sufficient condition for $\mathfrak{B}$ to be isomorphic to an endomorphic image of $\mathfrak{A}$ is that*

$$Th(\mathfrak{A}) \cap \Pi \subseteq Th(\mathfrak{B}) \quad \textit{and} \quad Th(\mathfrak{A}) \cap \Lambda \subseteq Th(\mathfrak{B}),$$

where Π is the set of all positive sentences and Λ is the set of all universal sentences.

Proof. The necessity is obvious.

To proof the sufficiency, let us observe that, because of $Th(\mathfrak{A}) \cap \Lambda \subseteq Th(\mathfrak{B})$ and by Theorem 2.1 of [2], $\mathfrak{B}$ is isomorphic to a substructure of $\mathfrak{A}$. Moreover, $Th(\mathfrak{A}) \cap \Pi \subseteq Th(\mathfrak{B})$, thus by Theorem 3.1 of [2], $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. So $\mathfrak{B}$ is isomorphic to an endomorphic image of $\mathfrak{A}$.

THEOREM 1.6. *Let ϑ be a sentence. Then the following conditions are equivalent*:

(i) *If ϑ holds in a structure $\mathfrak{C}$, then ϑ holds in every endomorphic image of $\mathfrak{C}$.*

(ii) *ϑ is logically equivalent to a conjunction of disjunctions of positive sentences and universal sentences.*

Now we will give a common generalization of the above Theorems. By operations we mean functions whose arguments are arbitrary algebraic structures and whose values are some classes of algebraic structures closed under isomorphisms and having the similarity type of the argument. For example, we often use the operations $\mathscr{H}$, $\mathscr{S}$ and $\mathscr{P}$ which are defined as follows:

$\mathscr{H}(\mathfrak{A})$ consists of all homomorphic images of $\mathfrak{A}$;

$\mathscr{S}(\mathfrak{A})$ consists of all structures isomorphic to substructures of $\mathfrak{A}$;

$\mathscr{P}(\mathfrak{A})$ consists of all structures isomorphic to direct powers of $\mathfrak{A}$.

Let $\mathscr{O}$ be such an operation. Then $\mathscr{O}^*$ denotes a certain inverse of $\mathscr{O}$, defined by

$$\mathfrak{B} \in \mathscr{O}^*(\mathfrak{A}) \textit{ if and only if } \mathfrak{A} \in \mathscr{O}(\mathfrak{B}).$$

Let $\Delta(\mathscr{O})$ denote the set of all sentences ϑ such that if ϑ holds in $\mathfrak{A}$, then ϑ holds in each $\mathfrak{B} \in \mathscr{O}(\mathfrak{A})$.

By these definitions we have immediately:

PROPOSITION 1.7. (i) *If* $\varphi, \psi \in \Delta(\mathcal{O})$, *then* $\varphi \wedge \psi$, $\varphi \vee \psi \in \Delta(\mathcal{O})$. *If* $\varphi \in \Delta(\mathcal{O})$ *and* $\vdash \varphi \leftrightarrow \chi$, *then* $\chi \in \Delta(\mathcal{O})$. *If* φ *is a sentence such that* $\vdash \varphi$ *or* $\vdash \neg\varphi$, *then* $\varphi \in \Delta(\mathcal{O})$.

(ii) $\Delta(\mathcal{O}^*) = \{\varphi : \neg\varphi \in \Delta(\mathcal{O})\}$.

Write $\mathfrak{A} \prec \mathfrak{B}$ if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$. An operation $\mathcal{O}$ is said to be *perfect* if and only if, for any structures $\mathfrak{A}$ and $\mathfrak{B}$, the condition

$$Th(\mathfrak{A}) \cap \Delta(\mathcal{O}) \subseteq Th(\mathfrak{B})$$

implies that there are structures $\mathfrak{A}_1 \succ \mathfrak{A}$ and $\mathfrak{B}_1 \succ \mathfrak{B}$ such that $\mathfrak{B}_1 \in \mathcal{O}(\mathfrak{A}_1)$.

Finally, an operation $\mathcal{O}$ is said to be *summable* if and only if for any two directed systems $\langle \mathfrak{A}_\delta \rangle_{\delta<\beta}$ and $\langle \mathfrak{B}_\delta \rangle_{\delta<\beta}$ (β being an ordinal >0) such that $\mathfrak{A}_{\delta_1} \prec \mathfrak{A}_{\delta_2}$ and $\mathfrak{B}_{\delta_1} \prec \mathfrak{B}_{\delta_2}$ for $\delta_1 < \delta_2 < \beta$ and $\mathfrak{B}_\delta \in \mathcal{O}(\mathfrak{A}_\delta)$ for all $\delta < \beta$, we have also

$$\bigcup_{\delta<\beta} \mathfrak{B}_\delta \in \mathcal{O}\Big(\bigcup_{\delta<\beta} \mathfrak{A}_\delta\Big).$$

Let $\langle \mathcal{O}_\gamma \rangle_{\gamma<\alpha}$ be a system of operations (α being an ordinal >0). Let τ be a similarity type, and let $\langle \tau_\gamma \rangle_{\gamma<\alpha}$ be a system of reducts of τ (3). For a structure $\mathfrak{A}$, we denote by $\mathfrak{A} | \tau_\gamma$ its τ_γ-reduct. Now we define an operation $\mathcal{O}$ as follows:

$$\mathfrak{B} \in \mathcal{O}(\mathfrak{A}) \textit{ if and only if } \mathfrak{B} | \tau_\gamma \in \mathcal{O}_\gamma(\mathfrak{A} | \tau_\gamma) \textit{ for all } \gamma < \alpha.$$

Such an operation $\mathcal{O}$ will be called a *composition* of $\langle \mathcal{O}_\gamma \rangle_{\gamma<\alpha}$ corresponding to the system $\langle \tau_\gamma \rangle_{\gamma<\alpha}$ of reducts.

THEOREM 1.8. *Let* τ *be a similarity type and* $\langle \tau_\gamma \rangle_{\gamma<\alpha}$ *a system of its reducts. Let* $\langle \mathcal{O}_\gamma \rangle_{\gamma<\alpha}$ *be a system of perfect summable operations. Then the composition* $\mathcal{O}$ *of* $\langle \mathcal{O}_\gamma \rangle_{\gamma<\alpha}$ *corresponding to the system* $\langle \tau_\gamma \rangle_{\gamma<\alpha}$ *is also perfect, and* $\Delta(\mathcal{O})$ *is the smallest set containing* $\bigcup_{\gamma<\alpha} \Delta(\mathcal{O}_\gamma)$ *and satisfying conditions* 1.7 (i).

Proof. Let Δ be the smallest set containing $\bigcup_{\gamma<\alpha} \Delta(\mathcal{O}_\gamma)$ and satisfying the conditions described in Proposition 1.7 (i). It is visible that $\Delta \subseteq \Delta(\mathcal{O})$. Thus, by Lemma 1.1. of [2], to prove our theorem it suffices to show that:

(*) *if* $Th(\mathfrak{A}) \cap \Delta \subseteq Th(\mathfrak{B})$, *then for some elementary extensions* $\mathfrak{A}^*$ *and* $\mathfrak{B}^*$ *of* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively, we have* $\mathfrak{B}^* \in \mathcal{O}(\mathfrak{A}^*)$.

To prove the existence of such $\mathfrak{A}^*$ and $\mathfrak{B}^*$ we define by induction two chains of structures $\mathfrak{A}_\gamma^{(n)}$ and $\mathfrak{B}_\gamma^{(n)}$ ($\gamma < \alpha$, $n < \omega$) having the following properties:

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(3) We recall that by a *similarity type* of a relational structure $\mathfrak{A} = \langle A, R_i \rangle_{i \in I}$ we mean the pair $\tau = \langle I, r \rangle$, where $r(i)$ is the number of places of $R_i (i \in I)$. By a *reduct* τ_0 of τ we mean the type $\langle I_0, r_0 \rangle$, where $I_0 \subseteq I$ and $r_0 = r|I_0$, and $\langle A, R_i \rangle_{i \in I_0}$ is the corresponding reduct of $\mathfrak{A}$.

1° *for all* $n < \omega$, *if* $\delta < \gamma < \alpha$, *then* $\mathfrak{A}_\delta^{(n)} \prec \mathfrak{A}_\gamma^{(n)}$ *and* $\mathfrak{B}_\delta^{(n)} \prec \mathfrak{B}_\gamma^{(n)}$;

2° $\mathfrak{A}_0^{(0)} = \mathfrak{A}$ *and* $\mathfrak{B}_0^{(0)} = \mathfrak{B}$; *moreover*, $\mathfrak{A}_0^{(n+1)} = \bigcup_{\gamma<\alpha} \mathfrak{A}_\gamma^{(n)}$ *and* $\mathfrak{B}_0^{(n+1)} = \bigcup_{\gamma<\alpha} \mathfrak{B}_\gamma^{(n)}$ *for* $n > 0$;

3° $\mathfrak{B}_\gamma^{(n)} | \tau_\gamma \in \mathcal{O}_\gamma(\mathfrak{A}_\gamma^{(n)} | \tau_\gamma)$ *for all* $n < \omega$ *and* $\gamma < \alpha$.

Let us put

$$\mathfrak{A}^* = \bigcup_{n<\omega} \bigcup_{\gamma<\alpha} \mathfrak{A}_\gamma^{(n)} \quad \text{and} \quad \mathfrak{B}^* = \bigcup_{n<\omega} \bigcup_{\gamma<\alpha} \mathfrak{B}_\gamma^{(n)}.$$

It is visible that $\mathfrak{A}^*$ and $\mathfrak{B}^*$ satisfy $(*)$ since both can be obtained as follows:

$$\mathfrak{A}^* = \bigcup_{n<\omega} \mathfrak{A}_\gamma^{(n)} \quad \text{and} \quad \mathfrak{B}^* = \bigcup_{n<\omega} \mathfrak{B}_\gamma^{(n)} \text{ for all } \gamma < \alpha.$$

Thus we have $\mathfrak{B}^* \in \mathcal{O}(\mathfrak{A}^*)$ which finishes the proof.

In the situations described in Theorems 1.2 and 1.4, τ_γ are partitions of τ and $\mathcal{O}_\gamma$ are isomorphisms in 1.2 and homomorphisms in 1.4. In the situation described in Theorem 1.6, we have $\tau_0 = \tau_1 = \tau$, $\mathcal{O}_0 = \mathscr{S}$ and $\mathcal{O}_1 = \mathscr{H}$.

2. Systems of operations on structures. Let $\langle \mathcal{O}_i \rangle_{i\in I}$ $(I \neq 0)$ be a system of operations and $\langle \mathfrak{A}_i \rangle_{i\in I}$ a system of similar structures. Let us denote by $\Delta\{\mathcal{O}_i; i\in I\}$ the set of all sentences ϑ such that if ϑ holds in all $\mathfrak{A}_i$ $(i\in I)$, then ϑ holds in every $\mathfrak{B} \in \bigcap_{i\in I} \mathcal{O}_i(\mathfrak{A}_i)$.

By the definition we immediately have

PROPOSITION 2.1. (i). $\bigcup_{i\in I} \Delta(\mathcal{O}_i) \subseteq \Delta\{\mathcal{O}_i : i\in I\}$.

(ii) *If* $\varphi, \psi \in \Delta\{\mathcal{O}_i : i\in I\}$ *and* $\vdash \varphi \leftrightarrow \chi$, *then* $\varphi \wedge \psi$, $\chi \in \Delta\{\mathcal{O}_i : i\in I\}$. *If* φ *is a sentence such that* $\vdash \varphi$ *or* $\vdash \neg\varphi$, *then* $\varphi \in \Delta\{\mathcal{O}_i : i\in I\}$.

Notice that $\Delta\{\mathcal{O}_i : i\in I\}$ needs not be closed under disjunction as we can show on the following example. Let $I = \{0, 1\}$, $\mathcal{O}_0 = \mathscr{H}$, $\mathcal{O}_1 = \mathscr{H}^*$, $\mathfrak{M}$ be a non-commutative semigroup without unity and $\mathfrak{N}$ a non-commutative semigroup with unity. Set $\mathfrak{A}_0 = \mathfrak{M} \times \mathfrak{N}$, $\mathfrak{B} = \mathfrak{N}$ and $\mathfrak{A}_1$ be an arbitrary commutative semigroup being a homomorphic image of $\mathfrak{B}$. Then $\mathfrak{B} \in \mathscr{H}(\mathfrak{A}_0)$ and $\mathfrak{B} \in \mathscr{H}^*(\mathfrak{A}_1)$. Let us consider the following sentences:

$$\varphi = \forall_x \forall_y [xy = yx] \quad \text{and} \quad \psi = \neg \exists_x \forall_y [xy = y].$$

It is easy to see that $\varphi, \psi \in \Delta\{\mathscr{H}, \mathscr{H}^*\}$, but $\varphi \vee \psi \notin \Delta\{\mathscr{H}, \mathscr{H}^*\}$. Indeed,

$$\mathfrak{A}_0 \models \varphi \vee \psi, \quad \mathfrak{A}_1 \models \varphi \vee \psi \quad \text{but} \quad \mathfrak{B} \models \neg\varphi \wedge \neg\psi.$$

Now we will study the structure of the sets $\Delta\{\mathcal{O}_i : i\in I\}$ for some systems of $\mathcal{O}_i$.

LEMMA 2.2. *Let* Δ_i *be the set of all sentences* ϑ *such that for each* $\mathfrak{A}_i$ *and* $\mathfrak{C}$ *with* $Th(\mathfrak{A}_i) \cap \Delta_i \subseteq Th(\mathfrak{C})$ *if* $\mathfrak{A}_i \models \vartheta$, *then also* $\mathfrak{C} \models \vartheta$ *(for all* $i\in I \neq 0$*). Let* Δ *be the smallest set of sentences containing* $\bigcup_{i\in I} \Delta_i$ *and satis-*

fying conditions described in 2.1 (ii). *Then a neccesary and sufficient condition for* $\vartheta \in \Delta$ *is that for each* $\mathfrak{A}_i$ *and* $\mathfrak{C}$ *such that:*

$$\Delta \cap \bigcap_{i \in I} Th(\mathfrak{A}_i) \subseteq Th(\mathfrak{C}) \tag{1}$$

and

$$\bigcup_{i \in I} \big(\Delta_i \cap Th(\mathfrak{A}_i)\big) \subseteq Th(\mathfrak{C}), \tag{2}$$

if $\mathfrak{A}_i \models \vartheta$ *for each* $i \in I$, *then also* $\mathfrak{C} \models \vartheta$.

Proof. The necessity is obvious. Let

$$\Gamma = \{\vartheta\colon (1),\ (2) \text{ and } A_i \models \vartheta\ (i \in I) \text{ imply } \mathfrak{C} \models \vartheta \text{ for all } \mathfrak{A}_i \text{ and } \mathfrak{C}\}.$$

Let us assume that $\vartheta \in \Gamma$. We will prove that $\vartheta \in \Delta$. If ϑ is inconsistent, then $\vartheta \in \Delta$ by 2.1 (ii). Thus we can assume that ϑ is consistent. Let

$$\Sigma = \{\sigma \in \Delta\colon \vdash \vartheta \to \sigma\}.$$

The set Σ is non-void since all tautologies are in Σ. Let $\mathfrak{C}$ be an arbitrary model of Σ and put

$$\Omega_i = \{\varphi\colon \neg\varphi \in \Delta_i \text{ and } \mathfrak{C} \models \varphi\} \quad \text{for all } i \in I.$$

Since Δ_i are closed under disjunctions, Ω_i are closed under conjunctions. Moreover, for each $i \in I$, the set $\Omega_i \cup \{\vartheta\}$ is consistent. Indeed, if $\Omega_i \cup \{\vartheta\}$ were inconsistent, then there were a $\varphi \in \Omega_i$ such that $\vdash \vartheta \to \neg\varphi$. But then we had $\neg\varphi \in \Delta_i \subseteq \Delta$ and thus $\neg\varphi \in \Sigma$. Since $\mathfrak{C}$ is a model of Σ, we have $\mathfrak{C} \models \neg\varphi$ contrary to our assumption that $\varphi \in \Omega_i$.

Let $\mathfrak{A}_i$ be an arbitrary model of $\Omega_i \cup \{\vartheta\}$. Then we have $\Delta \cap \bigcap_{i \in I} Th(\mathfrak{A}_i) \subseteq Th(\mathfrak{C})$. Indeed, if $\neg\varphi \in \Delta \cap \bigcap_{i \in I} Th(\mathfrak{A}_i)$, then there are sentences $\neg\varphi_k \in \Delta_{i_k}$ $(k = 1, \ldots, n)$ such that $\vdash \neg\varphi \leftrightarrow (\neg\varphi_1 \wedge \ldots \wedge \neg\varphi_n)$. We have either $\mathfrak{C} \models \varphi$ or $\mathfrak{C} \models \neg\varphi$. We eliminate the first possibility. Indeed, if $\mathfrak{C} \models \varphi$, then $\mathfrak{C} \models \varphi_1 \vee \ldots \vee \varphi_n$. Thus there exists a $k \in \{1, \ldots, n\}$ such that $\mathfrak{C} \models \varphi_k$. But $\neg\varphi_k \in \Delta_{i_k}$ and we have $\varphi_k \in \Omega_{i_k}$ which means that $\mathfrak{A}_{i_k} \models \varphi_k$. On the other hand, we have $\mathfrak{A}_{i_k} \models \neg\varphi$, thus $\mathfrak{A}_{i_k} \models \neg\varphi_k$; a contradiction. Since we have shown that φ cannot hold in $\mathfrak{C}$, thus $\mathfrak{C} \models \neg\varphi$ and, consequently, $\Delta \cap \bigcap_{i \in I} Th(\mathfrak{A}_i) \subseteq Th(\mathfrak{C})$.

Next, we prove that $\bigcup_{i \in I} \big(\Delta_i \cap Th(\mathfrak{A}_i)\big) \subseteq Th(\mathfrak{C})$. Indeed, let $\neg\varphi \in \bigcup_{i \in I} \big(\Delta_i \cap Th(\mathfrak{A}_i)\big)$. Then there is an index $i \in I$ such that $\neg\varphi \in \Delta_i \cap Th(\mathfrak{A}_i)$, thus $\neg\varphi \in \Delta_i$ and $\mathfrak{A}_i \models \neg\varphi$. Either φ or $\neg\varphi$ holds in $\mathfrak{C}$. We eliminate the first possibility. Indeed, if $\mathfrak{C} \models \varphi$, then $\varphi \in \Omega_i$, but $\mathfrak{A}_i$ is a model of Ω_i, whence $\mathfrak{A}_i \models \varphi$; a contradiction. Since φ cannot hold in $\mathfrak{C}$, we have $\mathfrak{C} \models \neg\varphi$ and, consequently, $\bigcup_{i \in I} \big(\Delta_i Th(\mathfrak{A}_i)\big) \subseteq Th(\mathfrak{C})$.

Now, since $\mathfrak{A}_i$ and $\mathfrak{C}$ satisfies (1) and (2), $\mathfrak{A}_i \models \vartheta$ for all $i \in I$, and $\vartheta \in \Gamma$, we have $\mathfrak{C} \models \vartheta$. This shows that ϑ holds in an arbitrary model of Σ. This means that there is a $\sigma \in \Sigma$ such that $\vdash \sigma \to \vartheta$, thus $\vdash \sigma \leftrightarrow \vartheta$ and $\vartheta \in \Delta$ by the definition of the set Δ, q.e.d.

Note that Lemma 2.2 is a multi-dimensional generalization of Lemma 1.1 in [2], and its proofs is similar.

THEOREM 2.3. *If* $\langle \mathcal{O}_i \rangle_{i \in I}$ $(I \neq 0)$ *is a system of perfect summable operations, then* $\Delta\{\mathcal{O}_i \colon i \in I\}$ *is the smallest set of sentences satisfying* 2.1 (i) *and* (ii).

Proof. Let Δ be the smallest set of sentences satisfying 2.1 (i) and (ii). The inclusion $\Delta\{\mathcal{O}_i \colon i \in I\} \supseteq \Delta$ follows from Proposition 2.1. Thus we must show that $\Delta\{\mathcal{O}_i \colon i \in I\} \subseteq \Delta$. Indeed, let $\vartheta \in \Delta\{\mathcal{O}_i \colon i \in I\}$ and let for a given system $\langle \mathfrak{A}_i \rangle_{i \in I}$ of structures and for a structure $\mathfrak{C}$, be $\mathfrak{A}_i \models \vartheta$ (for all $i \in I$) and

$$\Delta \cap \bigcap_{i \in I} Th(\mathfrak{A}_i) \subseteq Th(\mathfrak{C}) \quad \text{and} \quad \bigcup_{i \in I} \big(Th(\mathfrak{A}_i) \cap \Delta\big) \subseteq Th(\mathfrak{C}).$$

The second inclusion gives $Th(\mathfrak{A}_i) \cap \Delta(\mathcal{O}_i) \subseteq Th(\mathfrak{C})$, thus using perfectness and summability of $\mathcal{O}_i$ (in the same manner as in the proof of Theorem 1.8.) we obtain a system $\langle \mathfrak{A}_i^* \rangle_{i \in I}$ of structures and a structure $\mathfrak{C}^*$ such that:

$$\mathfrak{C}^* \succeq \mathfrak{C}, \quad \mathfrak{A}_i^* \succeq \mathfrak{A}_i \quad \text{and} \quad \mathfrak{C}^* \in \mathcal{O}_i(\mathfrak{A}_i^*) \quad \text{for all } i \in I.$$

$\mathfrak{C}^* \models \vartheta$ since $\vartheta \in \Delta\{\mathcal{O}_i \colon i \in I\}$ and thus we have $\mathfrak{C} \models \vartheta$. Hence, by Lemma 2.2, $\vartheta \in \Delta$, q.e.d.

From this theorem and well known theorems of Lyndon, Łoś and Tarski we immediately obtain the following corollaries characterizing $\Delta\{\mathcal{O}_i \colon i \in I\}$ in some cases considered by C. Ryll-Nardzewski which, as we have mentioned in the introduction, were proved by D. S. Scott.

COROLLARY 2.4. *For every sentence* ϑ *the following conditions are equivalent:*

(i) *If* $\mathfrak{A}_0 \models \vartheta$ *and* $\mathfrak{A}_1 \models \vartheta$, *and if there are homomorphisms of* $\mathfrak{A}_0$ *onto* $\mathfrak{B}$ *and* $\mathfrak{B}$ *onto* $\mathfrak{A}_1$, *then* $\mathfrak{B} \models \vartheta$.

(ii) ϑ *is logically equivalent to a conjunction of sentences which are positive or negative* (4).

COROLLARY 2.5. *For every sentence* ϑ *the following conditions are equivalent:*

(i) *If* $\mathfrak{A}_0 \models \vartheta$, $\mathfrak{A}_1 \models \vartheta$ *and* $\mathfrak{A}_0 \subseteq \mathfrak{B} \subseteq \mathfrak{A}_1$, *then* $\mathfrak{B} \models \vartheta$.

(ii) ϑ *is logically equivalent to a conjunction of existential or universal sentences.*

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(4) A sentence φ is negative if and only if $\neg \varphi$ is positive.

REFERENCES

[1] H. J. Keisler, *Ultraproducts and saturated models*, Indagationes Mathematicae 26 (1964), p. 178-186.

[2] — *Some applications of infinitely long formulas*, The Journal of Symbolic Logic 30 (1965), p. 339-349.

[3] M. Morley and L. R. Vaught, *Homogeneous-universal models*, Mathematica Scandinavica 11 (1962), p. 37-57.

[4] J. Mycielski, *Some compactifications of general algebras*, Colloquium Mathematicum 13 (1964), p. 1-9.

[5] — and C. Ryll-Nardzewski, *Equationally compact algebras (II)*, Fundamenta Mathematicae (in print).

[6] B. Węglorz, *Equationally compact algebras (I)*, ibidem 59 (1966), p. 263-272.

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