COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 2

COMPLETIONS OF NORMED LINEAR LATTICES

BY

TOGO NISHIURA (DETROIT, MICH.)

By a *normed linear lattice* $(L, \|\cdot\|, \leqslant)$ we mean a normed linear space $(L, \|\cdot\|)$ over the real field with a linear lattice structure $(L, \leqslant)$ whose order is compatible with the norm; i.e., $f_1, f_2 \in L$, $|f_1| \leqslant |f_2| \Rightarrow \|f_1\| \leqslant \|f_2\|$. As a normed linear space, L can be completed to NL with respect to the norm, and as an Archimedean linear lattice, it can be completed to OL with respect to the order. It is easy to show that the order can be extended to NL and the norm can be extended to OL so that normed linear lattices result.

We investigate the various iterations of O and N on normed linear lattices. It is shown that $NONL = ONL$; consequently, the iterations must terminate. A consequence of the above identity is

THEOREM A. *The order completion of a Banach lattice is a Banach lattice.*

This theorem may be known but its proof seems not to be available in the literature. We will also show that $ONL \neq NOL$ for some L.

Two problems are posed in this note.

1. Order completions. Let $(L, \leqslant)$ be a linear lattice. By an *order completion* of $(L, \leqslant)$ we mean a linear lattice $(OL, \leqslant)$ and a mapping $e: L \to OL$ which satisfy the following five conditions:

1. $(OL, \leqslant)$ is order complete.

2. e is a linear space isomorphism of L into OL.

3. If $f_1, f_2 \in L$, then $f_1 \leqslant f_2$ if and only if $e(f_1) \leqslant e(f_2)$.

4. If M is a non-empty subset of L bounded below by O and $\bigwedge M$ exists then $e(\bigwedge M) = \bigwedge e(M)$.

5. For each $g \in OL$,

$$\bigwedge \{e(f) \,|\, e(f) \geqslant g, f \in L\} = g = \bigvee \{e(f) \,|\, e(f) \leqslant g, f \in L\}.$$

If $(L, \leqslant)$ is an Archimedean linear lattice then such an order completion does exist (see [2], p. 140). Furthermore, all such completions are unique up to isomorphism. Since normed linear lattices are Archimedean, we have the existence of such completions in our setting.

Let $(L, \|\cdot\|, \leqslant)$ be a normed linear lattice and $(OL, \leqslant), e\colon L \to OL$ be the order completion. The norm $\|\cdot\|$ on L can be extended to a norm $\|\cdot\|$ on OL by the formula

$$\|g\| = \inf\{\|f\| \mid e(f) \geqslant |g|, f \in L\}, \qquad g \in OL.$$

It is clear that $\|\cdot\|$ is a pseudo norm on OL such that $g_1, g_2 \in OL$, $|g_1| \leqslant |g_2|$ imply $\|g_1\| \leqslant \|g_2\|$. To see it is a norm, we need only use conditions 5 and 3 above together with the fact that $(L, \|\cdot\|, \leqslant)$ is a normed linear lattice. It is clear that $e\colon L \to OL$ is an isometry with these norms. We have the following proposition:

PROPOSITION B. *If $(OL, |||\cdot|||, \leqslant)$ is another normed linear lattice so that $e\colon L \to OL$ is an isometry, then $|||\cdot||| \leqslant \|\cdot\|$.*

Proof. For each $g \in OL$,

$$|||g||| \leqslant \inf\{|||e(f)||| \mid e(f) \geqslant |g|, f \in L\} = \inf\{\|f\| \mid e(f) \geqslant |g|, f \in L\} = \|g\|.$$

In general, $(L, \|\cdot\|, \leqslant)$ as a normed linear space is not dense in $(OL, \|\cdot\|, \leqslant)$ (see example D below). The following problem on order completion presents itself:

PROBLEM 1. Suppose $(OL, |||\cdot|||, \leqslant)$ is a normed linear lattice so that $e\colon L \to OL$ is an isometry. Is it necessarily true that $|||\cdot||| = \|\cdot\|$? If the answer is negative, then is it possible to find a $(L, \|\cdot\|, \leqslant)$ and such a $(OL, |||\cdot|||, \leqslant)$ so that $(L, \|\cdot\|, \leqslant)$ is dense in $(OL, |||\cdot|||, \leqslant)$ and not dense in $(OL, \|\cdot\|, \leqslant)$ as normed linear spaces? (**P 641**)

Proof of Theorem A. Let $(L, \|\cdot\|, \leqslant)$ be a Banach lattice. We need to prove $(OL, \|\cdot\|, \leqslant)$ is norm complete. To this end, consider a sequence $\{g_n\}$ in OL such that $\|g_n - g_{n-1}\| \leqslant 2^{-n}$. We will show such a sequence converges.

Since $\|g_n - g_{n-1}\| \leqslant 2^{-n}$, there is $f_n \in L$ such that $|g_n - g_{n-1}| \leqslant e(f_n)$ and $\|f_n\| \leqslant 2^{-n+1}$. Let

$$r_n = \sum_{i=n+1}^{\infty} f_i.$$

r_n exists since L is a Banach space and $\|f_i\| \leqslant 2^{-i+1}$. Also,

$$\|r_n\| \leqslant 2^{-n+1} \quad \text{and} \quad r_n \geqslant \sum_{i=n+1}^{m} f_i \geqslant 0 \quad \text{for} \quad m > n.$$

Hence for $m > n$,

$$g_n - e(r_n) \leqslant g_n - e\Big(\sum_{i=n+1}^{m} f_i\Big) \leqslant g_m \leqslant g_n + e\Big(\sum_{i=n+1}^{m} f_i\Big) \leqslant g_n + e(r_n).$$

Since OL is order complete,

$$g_n - e(r_n) \leqslant \bigwedge_{m>n} g_m \leqslant \bigvee_{k=1}^{\infty} \bigwedge_{j>k} g_j \leqslant \bigwedge_{k=1}^{\infty} \bigvee_{j>k} g_j \leqslant \bigvee_{m>n} g_m \leqslant g_n + e(r_n).$$

Let

$$g = \bigvee_{k=1}^{\infty} \bigwedge_{j>k} g_j.$$

Then $|g - g_n| \leqslant e(r_n)$. Consequently, $\|g - g_n\| \leqslant 2^{-n+1}$. The proof is now completed.

2. Norm completions. Let $(L, \|\cdot\|)$ be a normed linear space. By a norm completion of $(L, \|\cdot\|)$ we mean a normed linear space $(NL, \|\cdot\|)$ and a mapping $e: L \to NL$ which satisfy the following conditions:

1. $(NL, \|\cdot\|)$ is a Banach space.
2. e is a linear isomorphism of L into NL.
3. $\|f\| = \|e(f)\|$ for every $f \in L$.
4. $e(L)$ is norm dense in NL.

The existence of norm completions is well known. Norm completions are unique up to isomorphisms.

If $(L, \|\cdot\|, \leqslant)$ is a normed linear lattice, then it is possible to define an order $\leqslant$ on $(NL, \|\cdot\|)$ so that $(NL, \|\cdot\|, \leqslant)$ is a normed linear lattice and $e: L \to NL$ is a linear lattice isomorphism of L into NL (see [1] or [3]). Since order is uniformly continuous with respect to the norm in normed linear lattices, the order $\leqslant$ on $(NL, \|\cdot\|)$ is uniquely determined by $(L, \|\cdot\|, \leqslant)$ if $(NL, \|\cdot\|, \leqslant)$ is to be a normed linear lattice.

In general, the linear lattice isomorphism $e: L \to NL$ is not order continuous. We give an example to illustrate this. Although the example may not be the simplest one, it is an amusing one.

EXAMPLE C. Let N be the set of natural numbers and $L = B(N)$ be the space of bounded sequences. We norm L by

$$\|\{x_n\}\| = \sum_{n=1}^{\infty} \frac{|x_n|}{n^2} + \limsup_{n\to\infty} |x_n|$$

and use the usual order. It is clear that L is order complete but not norm complete.

In order to describe the norm completion of L we give some notations. Let $\beta(N)$ be the Stone-Čech compactification of N and $Q = C[\beta(N) \setminus N]$ be the space of continuous functions on $\beta(N)\setminus N$ with the usual uniform norm $\|\cdot\|_\infty$. Next, let

$$P = \left\{\{x_n\} \,\middle|\, \sum_{n=1}^{\infty} \frac{|x_n|}{n^2} < \infty\right\}$$

with the obvious norm $\|\cdot\|_0$. Then $NL = P\oplus Q$ with norm $\|\{x_n\}+g\| = \|\{x_n\}\|_0+\|g\|_\infty$ and the usual order.

Consider now the sequences $\{x_n^i\}$, $i = 1, 2, \ldots$, where $x_n^i = 1$ if $n > i$ and $x_n^i = 0$ if $n \leqslant i$. Clearly,

$$e(\bigwedge_{i=1}^{\infty} \{x_n^i\}) = e(\{0\}) = \{0\}+0 \neq \{0\}+1 = \bigwedge_{i=1}^{\infty} e(\{x_n^i\}).$$

We now discuss our second problem. In Section 1 we found that $NONL = ONL$ for every L but the uniqueness of the norm on OL remained unresolved. In this section we found that the uniqueness of the order on NL is assured. It still remains unresolved as to whether or not $ONOL = NOL$ for every L. This is our second problem.

PROBLEM 2. Is it true that an order complete normed linear lattice $(L, \|\cdot\|, \leqslant)$ always yields an order complete normed linear lattice $(NL, \|\cdot\|, \leqslant)$ (**P 642**)?

3. Iterations of norm and order completions. By Theorem A, it is clear that any alternating iteration of order and norm completions on a normed linear lattice will produce no new normed linear lattices after a finite number of steps.

A natural problem is the commuting problem: Is it true that $(NOL, \|\cdot\|, \leqslant) = (ONL, \|\cdot\|, \leqslant)$? We show equality need not hold.

EXAMPLE D. Let

$$L = C[0,1], \qquad \|f\| = \int_0^1 |f|\,dx$$

and $\leqslant$ be the usual order. It is easy to see that $(ONL, \|\cdot\|, \leqslant)$ is the usual $L_1[0,1]$. We will show that $(NOL, \|\cdot\|, \leqslant)$ is not $L_1[0,1]$. To this end, we find $g_1, g_2 \in OL$ such that $g_1 \geqslant 0$, $g_2 \geqslant 0$ and $\|g_1+g_2\| < \|g_1\|+\|g_2\|$.

Let $U_1 \cup F \cup U_2 = [0,1]$, where U_1 and U_2 are two disjoint non-empty open sets, F is the boundary of each U_i $(i = 1, 2)$ and F is nowhere dense. Furthermore, let F have Lebesgue measure $m(F) > 0$. Denote by χ_i the characteristic function of the set U_i and define

$$g_i = \bigwedge\{e(f) | f \geqslant \chi_i, f \in L\} \qquad (i = 1, 2).$$

Let $\mathscr{L}(g_i) = \{f \in L \mid e(f) \leqslant g_i\}$. Since U_i is open, we have

$$\{f \in L \mid f \geqslant f', f' \in \mathscr{L}(g_i)\} = \{f \in L | f \geqslant \chi_i\}.$$

Now $g_1+g_2 = \bigwedge \{e(f) \,|\, f \geqslant f_1'+f_2', f_i' \in \mathscr{L}(g_i), i = 1, 2\}$. Also, since F is nowhere dense, it is an easy matter to show that $g_1+g_2 = e(1)$. Finally, that $g_1 \geqslant 0$, $g_2 \geqslant 0$, $\|g_1\| = 1-m(U_2)$ and $\|g_2\| = 1-m(U_1)$ are obvious. Hence,

$$\|g_1+g_2\| = e(1) = 1 < 1+m(F) = \|g_1\|+\|g_2\|.$$

Of course, until Problem 1 is settled, the commuting problem is not entirely resolved.

Acknowledgement. The author wishes to thank Professor Casper Goffman for bringing the iteration and commuting problems to his attention. Helpful conversations are also gratefully acknowledged.

REFERENCES

[1] M. M. Day, *Normed linear spaces*, Berlin 1962.
[2] H. Nakano, *Modern spectral theory*, Tokyo 1950.
[3] H. H. Shaefer, *Topological vector spaces*, New York 1966.

WAYNE STATE UNIVERSITY and PURDUE UNIVERSITY

Reçu par la Rédaction le 15. 3. 1967