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ON SYSTEMS OF MAPPINGS BETWEEN MODELS

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In this paper we consider a general existence theorem for models concerning α-systems of mappings between models. As a special case of this theorem we obtain the existence theorems from papers [2, 5, 7, 8, 10]. The main results of this paper were announced in [9].

§ 1. Relations and partial operations. Let A be an arbitrary set and let k be any ordinal number. By a *k-ary relation in the set A* we understand any subset of the set A^k of all sequences $(a_\xi, \xi < k)$ with $a_\xi \in A$ for $\xi < k$. Any mapping of any subset of the set A^k into the set A is said to be a *k-ary partial operation in the set A*. If f is a k-ary partial operation in the set A and if f is defined for a sequence $(a_\xi, \xi < k) \in A^k$, then by $f(a_\xi, \xi < k)$ will be denoted the value of f for $(a_\xi, \xi < k)$. The k-ary partial operations in the set A defined on the whole set A^k are called *k-ary operations in the set A*. Any k-ary partial operation f in the set A induces in A the $(k+1)$-ary relation $\mathrm{rel} f$ such that

$$(1) \qquad (a_\xi, \xi < k; a) \in \mathrm{rel} f \Leftrightarrow a = f(a_\xi, \xi < k).$$

The relation $\mathrm{rel}\, f = r$ has the following property:

(2) for each sequence $(a_\xi, \xi < k) \in A^k$ there exists at most one element $a \in A$ with $(a_\xi, \xi < k; a) \in r$.

For every $(k+1)$-ary relation r in the set A having property (2) there exists one and anly one k-ary partial operation f in A such that $r = \mathrm{rel}\, f$.

We define the notion of a *homomorphism of relations* as follows. Let r and s be two k-ary relations in the sets A and B, respectively. A mapping h of A into B is called a *homomorphism of r into s* if it has the following property:

(3) for all $(a_\xi, \xi < k) \in A^k$ the condition $(a_\xi, \xi < k) \in r$ implies the condition $\big(h(a_\xi), \xi < k\big) \in s$.

A homomorphism h of a k-ary relation r in a set A into a k-ary relation s in a set B is said to be *strong*, provided for each sequence $(a_\xi, \xi < k) \in A^k$, if $\big(h(a_\xi), \xi < k\big) \in s$, then there exists $a'_\xi \in A$ such that $h(a'_\xi) = h(a_\xi)$ for $\xi < k$ and $(a'_\xi, \xi < k) \in r$.

A mapping h of a set A into a set B is called a (*strong*) *homomorphism* of a k-ary partial operation f in the set A into a k-ary partial operation f' in the set B if h is a (strong) homomorphism of the relation $\operatorname{rel} f$ into the relation $\operatorname{rel} f'$. Let us observe that

(1.1) *A mapping h of a set A into a set B is a homomorphism of a k-ary partial operation f in the set A into a k-ary partial operation f' in the set B if and only if for each sequence $(a_\xi, \xi < k) \in A^k$ if f is defined for $(a_\xi, \xi < k)$, then f' is defined for $\big(h(a_\xi), \xi < k\big)$, and if, moreover, we have*

$$h\big(f(a_\xi, \xi < k)\big) = f'\big(h(a_\xi), \xi < k\big). \tag{4}$$

Proof. Let h be a homomorphism of f into f'. Then h is a homomorphism of the relation $\operatorname{rel} f$ into the relation $\operatorname{rel} f'$. Hence if $a = f(a_\xi, \xi < k)$, then, by (1), $(a_\xi, \xi < k; a) \in \operatorname{rel} f$, and thus, by (3), $\big(h(a_\xi), \xi < k, h(a)\big) \in \operatorname{rel} f'$. Therefore, by (1), $h(a) = f'\big(h(a_\xi), \xi < k\big)$, i.e. condition (4) is fulfied. Now let us assume that condition (4) holds. If $(a_\xi, \xi < k, a) \in \operatorname{rel} f$, then, by (1), $a = f(a_\xi, \xi < k)$, and thus, by (4), we have $h(a) = f'\big(h(a_\xi), \xi < k\big)$. Therefore, by (1), $\big(h(a_\xi), \xi < k; h(a)\big) \in \operatorname{rel} f'$, i.e. h is a homomorphism of $\operatorname{rel} f$ into $\operatorname{rel} f'$ and theorem (1.1) is proved.

The one-to-one homomorphisms are *isomorphisms*.

A subset B of a set A is said to be *algebraically closed with respect to a k-ary partial operation f in the set A*, provided for sequences $(b_\xi, \xi < k) \in B^k$, if f is defined for $(b_\xi, \xi < k)$, then the value $f(b_\xi, \xi < k)$ belongs to B.

We define the *direct product of relations* in the ordinary way. Let T be any set and let r_t, for $t \in T$, be any k-ary relation in the set A_t. Moreover, let $A = \mathop{\mathrm{P}}_{t\in T} A_t$ be the *Cartesian product of sets* A_t (i.e. A is the set of all mappings $\chi\colon T \to \bigcup_{t\in T} A_t$, with $\chi(t) \in A_t$ for $t \in T$). The *direct product of relations* r_t, where $t \in T$, is the k-ary relation r in the set $A = \mathop{\mathrm{P}}_{t\in T} A_t$ such that

$$(\chi_\xi, \xi < k) \in r \text{ if and only if } \big(\chi_\xi(t), \xi < k\big) \in r_t \text{ for } t \in T. \tag{5}$$

The *natural projection* p_t of A onto A_t with $p_t(\chi) = \chi(t)$ for $\chi \in A$ is a homomorphism of the relation r into the relation r_t for all $t \in T$. The *direct sum of the k-ary relations r_t in the sets A_t*, where $t \in T$, is defined as follows. The set of all pairs $\langle t, a\rangle$, where $t \in T$ and $a \in A_t$, which is called the *direct sum of sets A_t*, will be denoted by $\mathop{\mathrm{S}}_{t\in T} A_t$. The direct sum of relations $r_t, t \in T$, is the k-ary relation r' in the set $A = \mathop{\mathrm{S}}\limits_{t \in T} A_t$ such that

(6) $(\langle t_\xi, a_\xi \rangle, \xi < k) \in r'$ if and only if there exists an element $t^* \in T$ with $t_\xi = t^*$ for all $\xi < k$ and with $(a_\xi, \xi < k) \in r_{t^*}$.

The *natural injection* i_t of A_t into A' such that $i_t(a) = \langle t, a \rangle$ for $a \in A_t$ is a strong isomorphism of r_t into r'.

The k-ary partial operation f in the set $A = \mathop{\mathrm{P}}\limits_{t \in T} A_t$ is said to be the *direct product of the k-ary partial operations f_t in the sets $A_t, t \in T$*, provided the relation $\operatorname{rel} f$ is the direct product of the relations $\operatorname{rel} f_t\ t \in T$. The k-ary partial operation f' in the set $A' = \mathop{\mathrm{S}}\limits_{t \in T} A_t$ is called the *direct sum of k-ary partial operations f_t in the sets $A_t, t \in T$*, if the relation $\operatorname{rel} f'$ is the direct sum of the relations $\operatorname{rel} f_t, t \in T$. The direct product f and the direct sum f' of partial operations $f_t, t \in T$, always exist and are uniquely determined by $f_t, t \in T$.

Now we shall introduce the notions of *weak continuity* and *continuity* of partial operations in topological spaces. Let A be any topological Hausdorff space and let f be an arbitrary k-ary partial operation in A. The partial operation f is said to be *weakly continuous* provided, for every sequence $(a_\xi, \xi < k) \in A^k$, if f is defined for $(a_\xi, \xi < k)$ and $a = f(a_\xi, \xi < k)$, then for every *neighbourhood* V of the element a there exists a neighbourhood U of the sequence $(a_\xi, \xi < k)$ in the product space A^k such that $f(U) \subset V$, where $f(U)$ is the set of all values of f on the sequences belonging to U. Let f be a weakly continuous k-ary partial operation in the space A and let B be an algebraically closed subset B of A with respect to f. Then the *topological closure* $\bar{B}$ of B may be not algebraically closed with respect to f. A partial operation f in a topological space A is said to be *continuous* if f is weakly continuous and the topological closure of every algebraically closed subset with respect to f is also algebraically closed with respect to f. The notion of continuity and weak continuity for operations are identical.

§ 2. Models and quasi-algebras. Let F and R be two arbitrary sets of *operator* and of *relation symbols*. For each $f \in F$ and each $r \in R$ we shall denote by $n(f)$ and $n(r)$ the ordinal numbers n_1 and n_2 for which the operator symbol f is n_1-ary and the relation symbol r is n_2-ary. The numbers $n(f)$ and $n(r)$ are called the *ranks of f and of r*, respectively. Any system

(7) $$\mathbf{A} = \langle A, (f_{\mathbf{A}}) f \in F, (r_{\mathbf{A}}) r \in R \rangle$$

such that A is any set and $f_{\mathbf{A}}$ is an $n(f)$-ary partial operation in A for $f \in F$ and $r_{\mathbf{A}}$ is an $n(r)$-ary relation in A for $r \in R$, is called a *model of type* (F, R). The models of type $(F, \emptyset)$, where $\emptyset$ is an empty set, are said to be *quasi-algebras of type* F. If in a quasi-algebra of type F only operations appear, then it is called an *algebra of type* F. Let (7) be an arbitrary model of type (F, R) and let A' be a subset of A which is algebraically closed with respect to every partial operation f_A for $f \epsilon F$. Then A', and also the sequence $\boldsymbol{A}' = \langle A', (f_{A'})f \epsilon F, (r_{A'})r \epsilon R\rangle$ such that, for $f \epsilon F$ and $r \epsilon R$, $f_{A'} = f_A | A'$ and $r_{A'} = r_A | A'$ are the restriction of f_A and r_A to the subset A', are called *(algebraical) submodels of model* $\boldsymbol{A}$. The *intersection of submodels of* $\boldsymbol{A}$ is also a submodel $\boldsymbol{A}''$ of $\boldsymbol{A}$. Therefore for any subset B of a model $\boldsymbol{A}$ there exists the least submodel of $\boldsymbol{A}$ containing B, which is the intersection of all submodels $\boldsymbol{A}'$ of $\boldsymbol{A}$ with $B \subset A'$. We say that $\boldsymbol{A}''$ *is generated by* B and it is denoted by $C_{\boldsymbol{A}}B$. If $C_{\boldsymbol{A}}B = \boldsymbol{A}$, then B is said to be a *set of generators for model* $\boldsymbol{A}$. Let (7) and $\boldsymbol{B} = \langle B, (f_{\boldsymbol{B}})f \epsilon F, (r_{\boldsymbol{B}})r \epsilon R\rangle$ be two models of type (F, R). A mapping h of A into B is said to be a *(strong) homomorphism* of model $\boldsymbol{A}$ into model $\boldsymbol{B}$ if h is a (strong) homomorphism of $f_{\boldsymbol{A}}$ into $f_{\boldsymbol{B}}$ and of $r_{\boldsymbol{A}}$ into $r_{\boldsymbol{B}}$ for all $f \epsilon F$ and $r \epsilon R$.

It is easy to verify that

(2.1) *A one-to-one mapping i of A onto B is a strong isomorphism of model $\boldsymbol{A}$ onto model $\boldsymbol{B}$ if and only if the mappings i and i^{-1} are isomorphisms of $\boldsymbol{A}$ onto $\boldsymbol{B}$ and of $\boldsymbol{B}$ onto $\boldsymbol{A}$.*

Let T be any set and let $\boldsymbol{A}_t = \langle A_t, (f_{\boldsymbol{A}_t})f \epsilon F, (r_{\boldsymbol{A}_t})r \epsilon R\rangle$ be any model of type (F, R) for $t \epsilon T$. Then the model $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}})f \epsilon F, (r_{\boldsymbol{A}})r \epsilon R\rangle$ such that $A = \mathop{\mathrm{P}}_{t \epsilon T} A_t$ is the Cartesian product of sets A_t and, for all $f \epsilon F$ and all $r \epsilon R$, $f_{\boldsymbol{A}}$ and $r_{\boldsymbol{A}}$ are direct products of the partial operations $f_{\boldsymbol{A}_t}$ and of relations $r_{\boldsymbol{A}_t}$, where $t \epsilon T$, is called the *direct product of models* $\boldsymbol{A}_t$, $t \epsilon T$, and it is denoted by $\mathop{\mathrm{P}}_{t \epsilon T} \boldsymbol{A}_t$.

The model $\boldsymbol{A}' = \langle A', (f_{\boldsymbol{A}'})f \epsilon F, (r_{\boldsymbol{A}'})r \epsilon R\rangle$ such that $A' = \mathop{\mathrm{S}}_{t \epsilon T} A_t$ is a direct sum of sets A_t, and $f_{\boldsymbol{A}'}$ and $r_{\boldsymbol{A}'}$, for $f \epsilon F$ and for $r \epsilon R$, are direct sums of partial operations $f_{\boldsymbol{A}_t}$ and relations $r_{\boldsymbol{A}_t}$, where $t \epsilon T$, is called the *direct sum of models* $\boldsymbol{A}_t$, $t \epsilon T$, and it is denoted by $\mathop{\mathrm{S}}_{t \epsilon T} \boldsymbol{A}_t$.

Now we consider the notion of a *topological model*. Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}})f \epsilon F, (r_{\boldsymbol{A}})r \epsilon R\rangle$ be any model of type (F, R) and let A be a topological Hausdorff space such that for each $f \epsilon F$ the partial operation $f_{\boldsymbol{A}}$ is (weakly) continuous in the space A. Then $\boldsymbol{A}$ is said to be a *(weak) topological model of type* (F, R). Let $\boldsymbol{A}$ be an arbitrary (weak) topological model of type (F, R). By a *relative topological submodel of* $\boldsymbol{A}$ we understand any algebraical submodel of $\boldsymbol{A}$ with the induced topology. A relative topological submodel $\boldsymbol{A}'$ of $\boldsymbol{A}$ is called a *topological submodel of* $\boldsymbol{A}$ if A' is a topologically closed subset of the space A. The intersection of topological submodels of $\boldsymbol{A}$ is also a topological submodel of $\boldsymbol{A}$. Therefore for every subset B of A there exists the least topological submodel of $\boldsymbol{A}$

containing B, which is the intersection of all topological submodels $\boldsymbol{A}'$ of $\boldsymbol{A}$ with $B \subset A'$. This model will be denoted by $C^*_{\boldsymbol{A}}B$ and it is called *topologically generated by* B. Let us observe that

(2.2) $C^*_{\boldsymbol{A}}B = \overline{C_{\boldsymbol{A}}B}$ *for all topological models* $\boldsymbol{A}$ *and all subsets* B *of* A, *where* $\overline{C_{\boldsymbol{A}}B}$ *is the topological closure of the algebraical submodel* $C_{\boldsymbol{A}}B$ *of* $\boldsymbol{A}$ *generated by* B.

Theorem (2.2) is false for weak topological models. The notions of the weak topological and of the topological model of type (F, R) are identical only for models with operations. Every algebraical model $\boldsymbol{A}$ of type (F, R) may be considered as identical with the topological model $\boldsymbol{A}$ of type (F, R) with the discrete topology. From an algebraical model $\boldsymbol{A}$ one can obtain many topological models by introducing different topologies in the set A. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two (weak) topological models of type (F, R). An algebraical (strong) homomorphism h of $\boldsymbol{A}$ into $\boldsymbol{B}$ is said to be *topological* if h is a continuous mapping of the space A into the space B. A strong isomorphism i of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is called *topological* (resp. a *homeomorphism*) if the mappings i and i^{-1} are continuous.

If $\boldsymbol{A}_t$, for $t \in T$, is a (weak) topological model of type (F, R), then the direct product $\boldsymbol{A} = \mathop{\mathrm{P}}_{t \in T} \boldsymbol{A}_t$ and the direct sum $\boldsymbol{A}' = \mathop{\mathrm{S}}_{t\in T} \boldsymbol{A}_t$ of models $\boldsymbol{A}_t$, $t \in T$, may be considered as (weak) topological models of type (F, R) with the ordinary topologies; the *neighbourhoods of the space* $A = \mathop{\mathrm{P}}_{t\in T} A_t$ are the sets of the form $\bigcap_{t \in T'} p_t^{-1}(U_t)$, where T' is a finite subset of T and U_t is a neighbourhood in A_t and p_t is the natural projection of A onto A_t; the *neighbourhoods of the space* $A' = \mathop{\mathrm{S}}_{t\in T} A_t$ are the sets of the form $\{\langle t, a\rangle : a \in V\}$, where $t \in T$ and V is a neighbourhood in the space A_t.

§ 3. The direct product and the direct sum of systems of mappings. Let X and T be any sets. Moreover, let σ_t be any mapping of X into a set A_t, $t \in T$. The *direct product of mappings* σ_t, $t \in T$, is the mapping σ of X into the Cartesian product $A = \mathop{\mathrm{P}}_{t\in T} A_t$ of sets A_t such that for all $x \in X$ we have

$$\sigma(x) = \chi, \quad \text{where} \quad \chi(t) = \sigma_t(x) \text{ for each } t \in T. \tag{8}$$

Now, let us consider the mappings σ_t of a set A_t into a set X, where $t \in T$. *Then the direct sum of mappings* σ_t, $t \in T$, is the mapping σ' of the direct sum $A' = \mathop{\mathrm{S}}_{t\in T} A_t$ of the sets A_t into the set X such that

$$\sigma'(\langle t, a\rangle) = \sigma_t(a) \text{ for all } t \in T \text{ and all } a \in A_t. \tag{9}$$

Let $\alpha \geqslant 1$ be any ordinal number. By an *α-system of mappings of a set X into a set Y* we understand any sequence $\{\sigma_\mu, \mu < \alpha\}$, where σ_μ, for $\mu < a$, is a mapping of X into Y. Let $\sigma_t = \{\sigma_{t\mu}, \mu < a\}$, for $t \epsilon T$, be an a-system of mappings of a set X into a set A_t. Then the a-system $\sigma = \{\sigma_\mu, \mu < a\}$ of mappings of X into the set $A = \mathop{\mathrm{P}}_{t \epsilon T} A_t$ is said to be the *direct product of a-systems* $\sigma_t, t \epsilon T$, provided for every $\mu < a$ the mapping σ_μ is the direct product of mappings $\sigma_{t\mu}$ with $t \epsilon T$. We define the *direct sum of a-systems of mappings* as follows. Let $\psi_t = \{\chi_{t\mu}, \mu < a\}$, for $t \epsilon T$, be an a-system of mappings of a set A_t into a set X. Then the a-system $\chi = \{\chi_\mu, \mu < a\}$ of mappings of the set $A' = \mathop{\mathrm{S}}_{t \epsilon T} A_t$ into the set X such that, for every $\mu < a$, χ_μ is the direct sum of mappings $\chi_{t\mu}, t \epsilon T$, is called the *direct sum of the a-systems* $\chi_t, t \epsilon T$.

If X and Y are topological spaces and if χ is a continuous mapping of the space X into the space Y, then we say that χ is a *topological mapping of X into Y*. If $\sigma = \{\sigma_\mu, \mu < a\}$ is an a-system of topological mappings of X into Y, then we say that σ is a *topological a-system of mappings of X into Y*. Since the direct product (resp. the direct sum) of topological mappings is also a topological mapping, we have the next theorem:

(3.1) *The direct product* (*resp. the direct sum*) *of topological a-systems of mappings is also a topological a-system of mappings.*

§ 4. A general existence theorem for models. In this paragraph we shall consider the a-systems of mappings between models of type (F, R) and of type (F^*, R^*). Let $\boldsymbol{A} = \langle A, (f_{\boldsymbol{A}}) f \epsilon F, (r_{\boldsymbol{A}}) r \epsilon R \rangle$ be an arbitrary but fixed (topological) model of type (F, R) and let $\mathfrak{B}$ be any class of (topological) models of type (F^*, R^*). Any pair $(\sigma, \boldsymbol{B})$, where $\sigma = \{\sigma_\mu, \mu < a\}$ is a (topological) a-system of mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ (i.e. of A into B) and $\boldsymbol{B} \epsilon \mathfrak{B}$, is said to be a (*topological*) *a-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$*. Let $(\sigma, \boldsymbol{B})$ be a topological a-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ such that $\sigma = \{\sigma_\mu, \mu < a\}$, where for each $\mu < a$ the mapping σ_μ is a topological homeomorphism of the space A into the space B. Then we say that the pair $(\sigma, \boldsymbol{B})$ is a *topological a-system of $\mathfrak{B}$-extensions of $\boldsymbol{A}$*.

Now we introduce some relations between (topological) a-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$. Let $(\sigma, \boldsymbol{B})$ and $(\sigma', \boldsymbol{B}')$ be two (topological) a-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$. We say that:

1° $(\sigma, \boldsymbol{B}) \leqslant (\sigma', \boldsymbol{B}')$ if there exists exactly one (topological) homomorphism h of $\boldsymbol{B}$ into $\boldsymbol{B}'$ such that $\sigma' = h\sigma$ (i.e. $\sigma'_\mu = h\sigma_\mu$ for $\mu < a$, where $\sigma' = \{\sigma'_\mu, \mu < a\}$ and $\sigma = \{\sigma_\mu, \mu < a\}$);

2° $(\sigma, \boldsymbol{B}) \equiv (\sigma', \boldsymbol{B}')$ if there exists exactly one (topological) strong isomorphism h of $\boldsymbol{B}$ onto $\boldsymbol{B}'$ such that $\sigma' = h\sigma$.

Let Σ be any class of (topological) a-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$. A (topological) a-system $(\sigma, \boldsymbol{B})$ of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ is said to be (*topological*) *free in the class* Σ if $(\sigma, \boldsymbol{B}) \epsilon \Sigma$ and for every a-system $(\sigma', \boldsymbol{B}') \epsilon \Sigma$ we have $(\sigma, \boldsymbol{B}) \leqslant (\sigma', \boldsymbol{B}')$. Now we prove

(4.1) *A (topological) free a-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ, if it exists, is uniquely determined up to the relation $\equiv$.*

Proof. Let $(\sigma, \boldsymbol{B})$ and $(\sigma', \boldsymbol{B}')$ be two (topological) free a-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ. Then $\sigma' = h\sigma$ and $\sigma = h'\sigma'$, where h and h' are the (topological) homomorphisms of $\boldsymbol{B}$ into $\boldsymbol{B}'$ and of $\boldsymbol{B}'$ into $\boldsymbol{B}$. Hence we obtain $\sigma' = hh'\sigma'$ and $\sigma = h'h\sigma$. But we have also $\sigma' = I'\sigma'$ and $\sigma = I\sigma$, where I' and I are the identity isomorphisms of $\boldsymbol{B}'$ onto $\boldsymbol{B}'$ and of $\boldsymbol{B}$ onto $\boldsymbol{B}$. Thus, by 1°, we have $hh' = I'$ and $h'h = I$. Hence h and h' are one-to-one and "onto", and, moreover $h' = h^{-1}$. Therefore h is (topological) strong isomorphism of $\boldsymbol{B}$ onto $\boldsymbol{B}'$, whence $(\sigma, \boldsymbol{B}) \equiv (\sigma', \boldsymbol{B}')$, and thus (4.1) is proved.

(4.2) *Let $(\sigma, \boldsymbol{B})$ be a topological free a-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ. Then the model $\boldsymbol{A}$ has a topological a-system of $\mathfrak{B}$-extensions in the class Σ if and only if the pair $(\sigma, \boldsymbol{B})$ is a topological a-system of $\mathfrak{B}$-extensions of $\boldsymbol{A}$.*

Proof. Let $(\sigma', \boldsymbol{B}')$ be a topological a-system of $\mathfrak{B}$-extensions of $\boldsymbol{A}$ in the class Σ. Since $(\sigma, \boldsymbol{B})$ is a topological free a-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ, therefore $(\sigma, \boldsymbol{B}) \leqslant (\sigma', \boldsymbol{B}')$ and thus we have $\sigma' = h\sigma$, where h is some topological homomorphism of $\boldsymbol{B}$ into $\boldsymbol{B}'$. Hence it follows that $(\sigma, \boldsymbol{B})$ is also a topological a-system of $\mathfrak{B}$-extensions of $\boldsymbol{A}$, and the proof of (4.2) is finished.

Let $(\sigma, \boldsymbol{B})$ be an arbitrary (topological) a-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ and let $\boldsymbol{B}'$ be such a (topological, relative topological) submodel of $\boldsymbol{B}$ that the set $\sigma(A) = \bigcup_{\mu<a} \sigma_\mu(A)$, where $\sigma = \{\sigma_\mu, \mu < a\}$, is contained in $\boldsymbol{B}'$. Then the pair $(\sigma, \boldsymbol{B}')$ is said to be a (*topological, relative topological*) *subsystem of* $(\sigma, \boldsymbol{B})$. A class Σ of (topological) a-systems of $\mathfrak{B}$ mappings of $\boldsymbol{A}$ is called *quasi primitive* if the class Σ is closed with respect to the direct products and (topological) subsystems and if it is closed with respect to the relation $\equiv$.

Now we prove a general existence theorem for models.

THEOREM 1. *Let $\boldsymbol{A}$ be any topological model of type (F, R) and let $\mathfrak{B}$ be any class of topological models of type (F^*, R^*). Moreover, let Σ be an arbitrary quasi-primitive class of topological a-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$. Then there exists a topological free a-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ.*

Proof. Let $\mathfrak{m} = |A|\bar{a}$ (1) and let $\mathfrak{n}$ be a cardinal number such that for every model $\boldsymbol{B}$ of type (F^*, R^*) and for every subset X of B with $|X| \leqslant \mathfrak{m}$ the submodel $C_{\boldsymbol{B}}X$ of $\boldsymbol{B}$ generated by X has the power $\leqslant \mathfrak{n}$. Moreover, let $\mathfrak{m}^* = 2^{2^{\mathfrak{n}}}$. Then each topological Hausdorff space Y having

(1) By $|A|$ and $\bar{a}$, where A is any set and a is any ordinal number, we denote the powers of A and a, respectively.

a dense subset X with $|X| \leqslant \mathfrak{n}$ fulfils the relation $|Y| \leqslant \mathfrak{m}^*$. Let E be an arbitrary set with $|E| \geqslant \mathfrak{m}^*$ and let $\boldsymbol{B}$ be an arbitrary topological model of type (F^*, R^*) with $B \subset E$. Let us consider the family of models $\boldsymbol{B}_\lambda = \boldsymbol{B}$, where λ runs through the set $\alpha(\boldsymbol{A}, \boldsymbol{B})$ of all topological α-systems σ' of mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ such that $(\sigma', \boldsymbol{B}) \epsilon \Sigma$. Let $\sigma_{\boldsymbol{B}}$ be the direct product of all α-systems of mappings $\sigma' \epsilon \alpha(\boldsymbol{A}, \boldsymbol{B})$, i.e. $\sigma_{\boldsymbol{B}}$ is the unique α-system of mappings of $\boldsymbol{A}$ into $\boldsymbol{B}^{\alpha(\boldsymbol{A},\boldsymbol{B})}$ such that $p_\lambda \sigma_{\boldsymbol{B}} = \lambda$ for all $\lambda \epsilon \alpha(\boldsymbol{A}, \boldsymbol{B})$, where p_λ is the natural projection of $\boldsymbol{B}^{\alpha(\boldsymbol{A},\boldsymbol{B})}$ onto $\boldsymbol{B}_\lambda = \boldsymbol{B}$. Let σ be the direct product of α-systems of mappings $\sigma_{\boldsymbol{B}}$, where $B \subset E$ and $\boldsymbol{B}$ belongs to the class $r\Sigma$ of all models $\boldsymbol{B}$ for which there exists an α-system σ' with $(\sigma', \boldsymbol{B}) \epsilon \Sigma$. Then σ is the unique α-system of mappings of $\boldsymbol{A}$ into the direct product $\boldsymbol{B}_1 = \mathrm{P}\, \boldsymbol{B}^{\alpha(\boldsymbol{A},\boldsymbol{B})}$ of all direct powers $\boldsymbol{B}^{\alpha(\boldsymbol{A},\boldsymbol{B})}$ with $B \subset E$ and with $\boldsymbol{B} \epsilon r\Sigma$ such that $\sigma_{\boldsymbol{B}} = p_{\boldsymbol{B}} \sigma$, where $p_{\boldsymbol{B}}$ is the natural projection of $\boldsymbol{B}_1$ onto $\boldsymbol{B}^{\alpha(\boldsymbol{A},\boldsymbol{B})}$. Let $\boldsymbol{C}$ be the topological submodel of the topological model $\boldsymbol{B}_1 = \mathrm{P}\, \boldsymbol{B}^{\alpha(\boldsymbol{A},\boldsymbol{B})}$ generated by the set $\sigma(A) = \bigcup_{\mu<\alpha} \sigma_\mu(A)$, where $\sigma = \{\sigma_\mu, \mu < \alpha\}$, i.e. $\boldsymbol{C} = C^*_{\boldsymbol{B}_1} \sigma(A)$. Since Σ is quasi-primitive $(\sigma, \boldsymbol{C}) \epsilon \Sigma$. Now we prove that the pair $(\sigma, \boldsymbol{C})$ is a topological free α-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ. For this let $(\sigma', \boldsymbol{B}')$ be an arbitrary α-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ belonging to the class Σ. Let us denote by $\boldsymbol{D}$ the topological submodel of $\boldsymbol{B}'$ generated by the set $\sigma'(A) = \bigcup_{\mu<\alpha} \sigma'_\mu(A)$, where $\sigma' = \{\sigma'_\mu, \mu < \alpha\}$. Since $\boldsymbol{D}$ has a dense subset $C_{\boldsymbol{B}'} \sigma'(A)$ of the power $\leqslant \mathfrak{n}$, therefore $|\boldsymbol{D}| \leqslant \mathfrak{m}^*$ and thus there exists a topological model $\boldsymbol{B}$ of type (F^*, R^*) with $B \subset E$ such that $\boldsymbol{B}$ is topologically strongly isomorphic to $\boldsymbol{D}$. Let i be a topological strong isomorphism of $\boldsymbol{B}$ onto $\boldsymbol{D}$. Then we have $\sigma' = q \cdot \sigma$, where $q = i p_\lambda p_{\boldsymbol{B}} | \boldsymbol{C}$ with $\lambda = i^{-1} \sigma'$, and therefore $(\sigma, \boldsymbol{C}) \leqslant (\sigma', \boldsymbol{B}')$; the topological homomorphism q of $\boldsymbol{C}$ into $\boldsymbol{B}'$ is unique, since the set $\sigma(A)$ generates $\boldsymbol{C}$ topologically. Thus we have proved that the pair $(\sigma, \boldsymbol{C})$ is a topological free α-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ, i.e. Theorem 1 is proved.

THEOREM 2. *Let $\boldsymbol{A}$ be any topological model of type (F, R) and let $\mathfrak{B}$ be any class of topological models of type (F^*, R^*). Moreover, let Σ be an arbitrary quasi-primitive class of topological α-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ closed with respect to the relative topological subsystems and let $(\sigma, \boldsymbol{C})$ be the topological free α-system of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ which exists by Theorem 1. Then the model $\boldsymbol{C}$ is algebraically generated by the set $\sigma(A) = \bigcup_{\mu<\alpha} \sigma_\mu(A)$, where $\sigma = \{\sigma_\mu, \mu < \alpha\}$.*

Proof. Let $\boldsymbol{C}'$ be the submodel of $\boldsymbol{C}$ algebraically generated by the set $\sigma(A)$. Let us consider $\boldsymbol{C}'$ as a relative topological submodel of $\boldsymbol{C}$. Then, since Σ is closed with respect to the relative topological subsystems, the pair $(\sigma, \boldsymbol{C}')$ belongs to the class Σ and thus $(\sigma, \boldsymbol{C}) \leqslant (\sigma, \boldsymbol{C}')$. Hence there exists a unique topological homomorphism h of $\boldsymbol{C}$ into $\boldsymbol{C}'$ such that

$\sigma = h\sigma$. Thus h is the identity mapping on the set $\sigma(A) = \bigcup_{\mu<\alpha} \sigma_\mu(A)$ of generators for $\boldsymbol{C}'$, and, therefore, h maps $\boldsymbol{C}$ onto $\boldsymbol{C}'$ and h is the identity mapping on the whole set $\boldsymbol{C}'$. Since $\boldsymbol{C}'$ is dense in $\boldsymbol{C}$ and h is a continuous mapping of $\boldsymbol{C}$ onto $\boldsymbol{C}'$, therefore $\boldsymbol{C} = \boldsymbol{C}'$ and Theorem 2 is proved.

From Theorem 1 and Theorem 2 we obtain

THEOREM 3. *Let $\boldsymbol{A}$ be an arbitrary model of type (F, R) and let $\mathfrak{B}$ be any class of models of type (F^*, R^*). Moreover, let Σ be an arbitrary quasi-primitive class of α-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$. Then there exists a free α-system $(\sigma, \boldsymbol{C})$ of $\mathfrak{B}$-mappings of $\boldsymbol{A}$ in the class Σ and the model $\boldsymbol{C}$ is generated by the set $\sigma(A)$.*

Proof. We consider the algebraical models as topological with the discrete topology. In this way we obtain from the model $\boldsymbol{A}$ a topological model $\hat{\boldsymbol{A}}$, and from the classes $\mathfrak{B}$ and Σ we obtain the classes $\hat{\mathfrak{B}} = (\hat{\boldsymbol{B}}: \boldsymbol{B} \in \mathfrak{B})$ and $\hat{\Sigma} = \big((\sigma, \hat{\boldsymbol{B}}): (\sigma, \boldsymbol{B}) \in \Sigma\big)$. Obviously, the class $\hat{\Sigma}$ of topological α-systems of $\hat{\mathfrak{B}}$-mappings of $\hat{\boldsymbol{A}}$ is quasi-primitive and it is closed with respect to the relative topological subsystems, since all topologies are discrete. Thus by applying Theorem 1 and Theorem 2 for $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$ and $\hat{\Sigma}$ we obtain Theorem 3, and the proof of Theorem 3 is finished.

§ 5. Systems of mappings fulfilling the basic mapping-formulas. Let us denote by $L = L(F, R, X)$ and by $L^* = L(F^*, R^*. \Phi \times X)$ *the first-order open logic with the identity corresponding to models of type (F, R) and models of type (F^*, R^*) based on the set X and the set $\Phi \times X$ considered as the sets of variables.* Let $X = (x_\xi, \xi < \varrho^*)$ and $\Phi = (\varphi_\mu, \mu < \alpha)$. The pairs (φ_μ, x_ξ) will be also denoted by $\varphi_\mu(x_\xi)$. If a formula $p \in L$ (resp. $q \in L^*$) is generated by $(x_\mu, \xi < \beta)$ (resp. $\big(\varphi_\mu(x_\xi), \mu < \alpha, \xi < \beta\big)$), then we shall write $p = p(x_\xi, \xi < \beta)$ (resp. $q = q\big(\varphi_\mu(x_\xi), \mu < \alpha, \xi < \beta\big)$).

The formulas of the form

$$\text{(i)} \qquad p(x_\xi, \xi < \beta) \to q\big(\varphi_\mu(x_\xi), \mu < \alpha, \xi < \beta\big),$$

where $p \in L$ and $q \in L^*$, will be called *basic mapping-formulas of type $(F, R; F^*, R^*)$*. Let $\sigma = \{\sigma_\mu, \mu < \alpha\}$ be an arbitrary (topological) α-system of mappings of a (topological) model $\boldsymbol{A}$ of type (F, R) into a (topological) model $\boldsymbol{B}$ of type (F^*, R^*).

We say that the *system $\sigma = \{\sigma_\mu, \mu < \alpha\}$ fulfils the basic mapping-formula* (i) provided that for each sequence $(a_\xi, \xi < \beta) \in A^\beta$ if the relation $p(a_\xi, \xi < \beta)$ holds in the model $\boldsymbol{A}$, then the relation $q\big(\sigma_\mu(a_\xi), \mu < \alpha, \xi < \beta\big)$ holds in the model $\boldsymbol{B}$.

Let P be any set of basic mapping-formulas (see (i)) and let $(\sigma, \boldsymbol{B})$ be a (topological) α-systems of $\mathfrak{B}$-mappings, where $\mathfrak{B}$ is any class of (topological) models of type (F^*, R^*), of a (topological) model $\boldsymbol{A}$ of type (F, R) and let the α-system σ fulfil every basic mapping formula belonging to the set P. Then the pair $(\sigma, \boldsymbol{B})$ is said to be a *(topological) a-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$*. A (topological) free a-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ in the class Λ of all (topological) a-systems of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ is called a *(topological) $\mathfrak{B}$-free a-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$*. If the pair $(\sigma, \boldsymbol{B})$ is the (topological) $\mathfrak{B}$-free a-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$, then the model $\boldsymbol{B}$ is called *(topological) a-P-free determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$*, and the a-system σ — a *canonical a-system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$*.

The model $\boldsymbol{B}$ (topological) a-P-free determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$, if it exists, is uniquely determined up to (topological) strong isomorphisms. The (topological) a-$\emptyset$-free model $\boldsymbol{B}$ determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$, where $\emptyset$ is the empty set of basic mapping-formulas, is said to *be (topological) a-free determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$*. Now we consider the problem of the existence of a (topological) a-P-free model determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$. The existence of this model depends on the set P and the class $\mathfrak{B}$. A formula $q = q\big(\varphi_\mu(x_\xi), \mu < a, \xi < \beta\big)$ in L^* is called *productable* provided for any set T and any models $\boldsymbol{B}_t, t \epsilon T$, of type (F^*, R^*) and for any family $\sigma_t = \{\sigma_{t\mu}, \mu < a\}$, $t \epsilon T$, of a-systems of mappings of X into B_t, if the relation $q\big(\sigma_{t\mu}(x_\xi), \mu < a, \xi < \beta\big)$ holds in the model $\boldsymbol{B}_t$ for all $t \epsilon T$, then the relation $q\big(\sigma_\mu(x_\xi), \mu < a, \xi < \beta\big)$ holds in the model $\boldsymbol{B}$, where $\boldsymbol{B} = \mathop{\mathrm{P}}_{t \epsilon T} \boldsymbol{B}_t$ and the a-system $\sigma = \{\sigma_\mu, \mu < a\}$ is the direct product of a-systems $\sigma_t, t \epsilon T$. Let us oberve that

(5.1) *The atomic formulas of logic L^* are productable.*

(5.2) *The conjunction of productable formulas is also a productable formula.*

A basic mapping-formula (i) is said to be *productable* if the formula $q\big(\varphi_\mu(x_\xi), \mu < a, \xi < \beta\big)$ is productable. A class $\mathfrak{B}$ of (topological) models of type (F^*, R^*) is called *quasi-primitive* if the class $\mathfrak{B}$ is closed with respect to the direct products of models, (relative topological) submodels of models and with respect to (topological) strong isomorphisms. From Theorem 1 and 2 we obtain

THEOREM 4. *Let $\boldsymbol{A}$ be an arbitrary topological model of type (F, R) and let $\mathfrak{B}$ be any quasi-primitive class of topological models of type (F^*, R^*). Moreover, let P be any set (possibly empty) of productable basic mapping-formulas of type $(F, R; F^*, R^*)$. Then there exists a topological $\mathfrak{B}$-free a-system $(\sigma, \boldsymbol{C})$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$. Moreover, the model $\boldsymbol{C}$ is algebraically generated by the set $\sigma(A)$.*

Proof. The class Σ of all topological a-systems $(\sigma', \boldsymbol{B}')$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ is quasi-primitive and closed with respect to relative topological subsystems, since $\mathfrak{B}$ is quasi-primitive and P is a set of productable basic mapping-formulas (resp. P is emtpy). Hence we obtain Theorem 4 from Theorem 1 and 2, which ends the proof.

From Theorem 4 immediately follows

(5.3) *For every topological model* $\boldsymbol{A}$ *of type* (F, R), *for every quasi-primitive class* $\mathfrak{B}$ *of topological model of type* (F^*, R^*) *and for every set* P *of productable basic mapping-formulas of type* $(F, R; F^*, R^*)$ *there exists a topological* a-P-*free model* $\boldsymbol{C}$ *determined by* $\boldsymbol{A}$ *in the class* $\mathfrak{B}$.

The model $\boldsymbol{C}$ *is algebraically generated by* $\sigma(A)$, *where* σ *is the canonical* a-*system of* P-*mappings of* $\boldsymbol{A}$ *into* $\boldsymbol{C}$.

Theorem 3 implies

THEOREM 5. *Let* $\boldsymbol{A}$ *be an arbitrary model of type* (F, R) *and let* $\mathfrak{B}$ *be any quasi-primitive class of models of type* (F^*, R^*). *Moreover, let* P *be any set (possibly empty) of productable basic mapping-formulas of type* $(F, R; F^*, R^*)$. *Then there exists a* $\mathfrak{B}$-*free* a-*system* $(\sigma, \boldsymbol{C})$ *of* $\mathfrak{B}$-P-*mappings of* $\boldsymbol{A}$. *The model* $\boldsymbol{C}$ *is generated by* $\sigma(A)$.

Proof. The class Σ of all a-systems of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ is quasi-primitive, since $\mathfrak{B}$ is quasi-primitive and P is a set of productable basic mapping-formulas. Hence Theorem 5 follows from Theorem 3.

From Theorem 5 immediately follows

(5.4) *Let* $\boldsymbol{A}$ *be any model of type* (F, R) *and let* P *be any set of productable basic mapping-formulas of type* $(F, R; F^*, R^*)$. *Then for every quasi-primitive class* $\mathfrak{B}$ *of models of type* (F^*, R^*) *there exists an* a-P-*free model* $\boldsymbol{C}$ *determined by* $\boldsymbol{A}$ *in the class* $\mathfrak{B}$. *The model* $\boldsymbol{C}$ *is generated by* $\sigma(A)$, *where* σ *is the canonical* a-*system of* P-*mappings of* $\boldsymbol{A}$ *into* $\boldsymbol{C}$.

Let $P = \emptyset$ be the empty set. Then the empty set $P = \emptyset$ may be considered also as a set of productable basic mapping-formulas and the a-systems of $\mathfrak{B}$-$\emptyset$-mappings of $\boldsymbol{A}$ are identical with the a-systems of $\mathfrak{B}$-mappings of $\boldsymbol{A}$. Thus from Theorem 4, (5.3), Theorem 5 and from (5.4) for $P = \emptyset$ immediately follow the following theorems:

(5.5) *Let* $\boldsymbol{A}$ *be any topological model of type* (F, R) *and let* $\mathfrak{B}$ *be any quasi-primitive class of topological models of type* (F^*, R^*). *Then there exists a topological* $\mathfrak{B}$-*free* a-*system* $(\sigma, \boldsymbol{C})$ *of* $\mathfrak{B}$-*mappings of* $\boldsymbol{A}$. *The model* $\boldsymbol{C}$ *is algebraically generated by* $\sigma(A)$.

(5.6) *For every topological model* $\boldsymbol{A}$ *of type* (F, R) *and for every quasi-primitive class* $\mathfrak{B}$ *of models of type* (F^*, R^*) *there exists a topological* a-*free model* $\boldsymbol{C}$ *determined by* $\boldsymbol{A}$ *in the class* $\mathfrak{B}$. *The model* $\boldsymbol{C}$ *is algebraically generated by* $\sigma(A)$, *where* σ *is the canonical* a-*system of mappings of* $\boldsymbol{A}$ *into* $\boldsymbol{C}$.

(5.7) *Let* $\boldsymbol{A}$ *be any model of type* (F, R) *and let* $\mathfrak{B}$ *be any quasi-primitive class of models of type* (F^*, R^*). *Then there exists a* $\mathfrak{B}$-*free* a-*system* $(\sigma, \boldsymbol{C})$ *of* $\mathfrak{B}$-*mappings of* $\boldsymbol{A}$. *The model* $\boldsymbol{C}$ *is generated by the set* $\sigma(A)$.

(5.8) *For every model* $\boldsymbol{A}$ *of type* (F, R) *and for every quasi-primitive class* $\mathfrak{B}$ *of models of type* (F^*, R^*) *there exists an* a-*free model* $\boldsymbol{C}$ *determined by* $\boldsymbol{A}$ *in the class* $\mathfrak{B}$. *The model* $\boldsymbol{C}$ *is generated by the set* $\sigma(A)$, *where* σ *is the canonical* a-*system of mappings of* $\boldsymbol{A}$ *into* $\boldsymbol{C}$.

§ 6. The common α-systems of $\mathfrak{B}$-P-mappings. Let T be any set and let $\boldsymbol{A}_t$, for $t \in T$, be a (topological) model of type (F, R) and let $\mathfrak{B}$ be any class of (topological) models of type (F^*, R^*).

Moreover, let P be an arbitrary set of basic mapping-formulas of type $(F, R; F^*, R^*)$. A pair $(\Psi, \boldsymbol{B})$, where $\boldsymbol{B} \in \mathfrak{B}$ and $\Psi = \{\sigma_t, t \in T\}$ with $\sigma_t = \{\sigma_{t\mu}, \mu < \alpha\}$ for $t \in T$ are any topological α-systems of P-mappings (i.e. mappings fulfiling every basic mapping-formula of set P) of (topological) model $\boldsymbol{A}_t$ of type (F, R) into $\boldsymbol{B}$, is said to be a *(topological) common α-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}_t$, $t \in T$*. Let $(\Psi, \boldsymbol{B})$ and $(\Psi', \boldsymbol{B}')$ be two (topological) common α-systems of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}_t, t \in T$. Then we say that

1° $(\Psi, \boldsymbol{B}) \leqslant (\Psi', \boldsymbol{B}')$ if there exists exactly one (topological) homomorphism h of $\boldsymbol{B}$ into $\boldsymbol{B}'$ such that $\Psi' = h\Psi$, i.e. $\sigma_t' = h\sigma_t$ for $t \in T$;

2° $(\Psi, \boldsymbol{B}) \equiv (\Psi', \boldsymbol{B}')$ if there exists exactly one (topological) strong isomorphism h of $\boldsymbol{B}$ onto $\boldsymbol{B}'$ such that $\Psi' = h\Psi$.

A (topological) common α-system $(\Psi, \boldsymbol{B})$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}_t$, $t \in T$, is called *(topological) $\mathfrak{B}$-free* if for every (topological) common α-system $(\Psi', \boldsymbol{B}')$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}_t, t \in T$, we have $(\Psi, \boldsymbol{B}) \leqslant (\Psi', \boldsymbol{B}')$.

Now we prove

THEOREM 6. *For any family $\boldsymbol{A}_t, t \in T$, of topological models of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of (topological) models of type (F^*, R^*) and for an arbitrary set P of productable basic mapping-formulas of type $(F, R; F^*, R^*)$ there exists a topological $\mathfrak{B}$-free common α-system $(\Psi, \boldsymbol{B})$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}_t, t \in T$.*

Proof. Let $\boldsymbol{A} = \underset{t \in T}{\mathrm{S}} \boldsymbol{A}_t$ be the direct sum of models $\boldsymbol{A}_t, t \in T$. Let $(\sigma, \boldsymbol{B})$ be the topological $\mathfrak{B}$-free α-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$, which exists by Theorem 4. Let $\sigma_t = i_t \sigma$, where i_t is the natural injection of $\boldsymbol{A}_t$ into $\boldsymbol{A}$. Then the pair $(\Psi, \boldsymbol{B})$, where $\Psi = \{\sigma_t, t \in T\}$ is the topological common $\mathfrak{B}$-free α-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}_t, t \in T$. The proof of Theorem 6 is thus finished.

The topological model $\boldsymbol{B}$ from Theorem 6 is said to be a *topological α-$\mathfrak{B}$-P-direct sum of $\boldsymbol{A}_t, t \in T$* and it is denoted by α-$\mathfrak{B}$-P-$\underset{t \in T}{\mathrm{S}} \boldsymbol{A}_t$. The topological α-$\mathfrak{B}$-P-direct sum of $\boldsymbol{A}_t$, $t \in T$, is uniquely determined up to topological strong isomorphisms and it exists for any quasi-primitive class $\mathfrak{B}$ of topological models of type (F^*, R^*) and for any set P of productable basic mapping-formulas of type $(F, R; F^*, R^*)$.

If we admit only the discrete topologies, then from Theorem 6 we immediately obtain

THEOREM 7. *For any family $\boldsymbol{A}_t, t \in T$, of models of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of models of type (F^*, R^*) and for an arbitrary set P of productable basic mapping-formulas of type $(F, R; F^*, R^*)$ there exists a $\mathfrak{B}$-free common α-system $(\Psi, \boldsymbol{B})$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}_t, t \in T$.*

The model $\boldsymbol{B}$ from Theorem 7 is called an *α-$\mathfrak{B}$-P-direct sum of models $A_t, t\in T$* and it is uniquely determined up to strong isomorphisms. The α-$\mathfrak{B}$-P-direct sums of models $\boldsymbol{A}_t, t\in T$, exists for any quasi-primitive class $\mathfrak{B}$ of models of type (F^*, R^*) and for every set P of productable basic mapping-formulas of type $(F, R; F^*, R^*)$.

§ 7. Special cases. Now we consider some special cases.

A. *Q-mappings.* Let Q be any set of basic mapping-formulas (i) of type $(F, R; F^*, R^*)$, where $q(\varphi_\mu(x_\xi), \mu<\alpha, \xi<\beta)$ is a conjunction of atomic formulas of the logic $L^* = L(F^*, R^*, \Phi\times X)$. Then, by (5.2), Q is a set of productable basic mapping-formulas and thus from Theorems 4, 5, 6, 7, (5.3) and (5.4) we immediately obtain the following theorems:

(7.1) *For any (topological) model $\boldsymbol{A}$ of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of (topological) models of type (F^*, R^*) there exists a (topological) $\mathfrak{B}$-free α-system $(\sigma, \boldsymbol{C})$ of $\mathfrak{B}$-Q-mappings of $\boldsymbol{A}$. The model $\boldsymbol{C}$ is algebraically generated by the set $\sigma(A)$.*

(7.2) *For any (topological) model $\boldsymbol{A}$ of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of (topological) models of type (F^*, R^*) there exists a (topological) α-Q-free model $\boldsymbol{C}$ determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$. The model $\boldsymbol{C}$ is algebraically generated by $\sigma(A)$, where σ is the canonical α-system of Q-mappings of $\boldsymbol{A}$ into $\boldsymbol{C}$.*

(7.3) *For any family $\boldsymbol{A}_t, t\in T$, of (topological) models of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of (topological) models of type (F^*, R^*) there exists a (topological) $\mathfrak{B}$-free common α-system of $\mathfrak{B}$-Q-mappings of $\boldsymbol{A}_t, t\in T$, and thus there exists a (topological) α-$\mathfrak{B}$-Q-direct sum of $\boldsymbol{A}_t, t\in T$.*

From Theorem (7.3) for $R = R^* = \emptyset$ and Q being the set of basic mapping-formulas $(\mathrm{i}_{f\varrho})$ (see § 8, section C) the existence theorems of papers [8] and [10] follow.

B. *$\mathfrak{B}$-homomorphisms of models.* In this section we shall assume that $F = F^*$ and $R = R^*$. When considering mappings between (topological) models of the same type, we often admit only mappings that are (topological) homomorphisms. A (topological) α-system $\sigma = \{\sigma_\mu, \mu<\alpha\}$ of mappings of a (topological) model $\boldsymbol{A}$ of type (F, R) into a (topological) model $\boldsymbol{B}$ of type (F, R) is a (topological) α-system of homomorphisms of A into B if and only if the set H of basic mapping-formulas

$$(\mathrm{i}_1)\qquad x = f(x_\xi, \xi<n(f)) \to \varphi_\mu(x) = f(\varphi_\mu(x_\xi), \xi<n(f)),$$

$$(\mathrm{i}_2)\qquad r(x_\xi, \xi<n(f)) \to r(\varphi_\mu(x_\xi), \xi<n(r))$$

of type $(F, R; F, R)$, where $f\in F$, $r\in R$ and $\mu<\alpha$, is fulfilled by the α-system σ. Hence the (topological) α-systems of H-mappings are ordinary (topological) α-systems of homomorphisms. The set H is a set of productable basic mapping-formulas of type $(F, R; F, R)$. Let P be an arbitrary set of basic mapping-formulas of type $(F, R; F, R)$. Then the (topological) a-systems of $P \cup H$-mappings of a (topological) model $\boldsymbol{A}$ of type (F, R) into a (topological) model $\boldsymbol{B}$ of type (F, R) are called *(topological) a-systems of P-homomorphisms of $\boldsymbol{A}$ into $\boldsymbol{B}$* (2). Moreover, the (topological) a-systems $(\sigma, \boldsymbol{B})$ of $\mathfrak{B}$-$P \cup H$-mappings of a (topological model) $\boldsymbol{A}$ of type (F, R), where $\mathfrak{B}$ is an arbitrary class of topological models of type (F, R), will be called *the (topological) a-systems of $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$.*

Since H is a set of productable basic mapping-formulas of type $(F, R: F, R)$, from Theorem 4 (5.3), Theorem 5, (5.4) and from Theorem 6 and 7 immediately follow the next theorems of this section:

(7.4) *For a (topological) model $\boldsymbol{A}$ of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of (topological) models of type (F, R) and for an arbitrary set P of productable basic mapping-formulas of type $(F, R; F, R)$ there exists a (topological) $\mathfrak{B}$-free a-system $(\sigma, \boldsymbol{C})$ of $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}$. The model $\boldsymbol{C}$ is algebraically generated by $\sigma(A)$.*

The model $\boldsymbol{C}$ from (7.4) is the (topological) a-$P \cup H$-free model determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$ and it will be also called the *(topological) a-P-free with respect to homomorphisms determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$. Then we have*

(7.5) *For any (topological) model $\boldsymbol{A}$ of type (F, R) and for any quasi primitive class $\mathfrak{B}$ of (topological) models of type (F, R) and for an arbitrary set P of productable basic mapping-formulas of type $(F, R; F, R)$ there exists a (topological) model a-P-free with respect to homomorphisms $\boldsymbol{C}$ determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$. The model $\boldsymbol{C}$ is algebraically generated by the set $\sigma(A)$, where σ is the canonical a-system of P-homomorphisms of $\boldsymbol{A}$ into $\boldsymbol{C}$.*

(7.6) *For any family $\boldsymbol{A}_t$, $t \in T$, of (topological) models of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of (topological) models of type (F, R) and for an arbitrary set P of productable basic mapping-formulas of type $(F, R; F, R)$ there exists a (topological) $\mathfrak{B}$-free common a-system $(\Psi, \boldsymbol{B})$ of $\mathfrak{B}$-P-homomorphisms of $\boldsymbol{A}_t$, $t \in T$.*

The (topological) model $\boldsymbol{B}$ from (7.6) is the (topological) a-$\mathfrak{B}$-$P \cup H$--direct sum of $\boldsymbol{A}_t$, $t \in T$, and it will be also called the *(topological) a-$\mathfrak{B}$-P-direct sum of $\boldsymbol{A}_t$, $t \in T$, with respect to (topological) homomorphisms.* Then we have

(2) Let us observe that the P-homomorphisms in the sense of this paper are ordinary homomorphisms fulfilling a set P of basic mapping-formulas. The P-homomorphisms in the sense of paper [8] are not ordinary homomorphisms, but they are some Q-mappings, where Q is a set of basic mapping-formulas of the form $(\mathrm{i}_{f\varrho})$ (see § 2, Section C), since in paper [8] we have denoted by P a $P_{F,G}$-mapping which induces a set Q of basic mapping-formulas of the form $(\mathrm{i}_{f\varrho})$.

(7.7) *For any family A_t, $t \in T$, of (topological) models of type (F, R) and for any quasi-primitive class $\mathfrak{B}$ of (topological) models of type (F, R) and for an arbitrary set P of productable basic mapping-formulas of type $(F, R; F, R)$ there exists a (topological) α-$\mathfrak{B}$-P-direct sum of A_t, $t \in T$, with respect to (topological) homomorphisms.*

Putting $R = \emptyset$ in (7.4) and (7.6), where $\emptyset$ is the empty set, and assuming all topologies to be discrete, we obtain the existence theorems contained in paper [5] of Schmidt. From the above theorems for $R = \emptyset$ and $P = \emptyset$ also follow the existence theorems of paper [7].

C. *The α-systems of mappings of topological spaces.* Now we shall consider the case $F = \emptyset$ and $R = \emptyset$. Let X be an arbitrary topological Hausdorff space. In the sequel, we shall consider X as a topological model of type $(\emptyset, \emptyset)$. Let

$$q\big(\varphi_\mu(x_\xi),\ \mu < \alpha,\ \xi < \beta\big) \tag{ii}$$

be an arbitrary formula of the logic $L^* = L(F^*, R^*, \Phi \times X)$. Moreover, let $\sigma = \{\sigma_\mu, \mu < \alpha\}$ be any topological α-system of mappings of X into a topological model $\boldsymbol{B}$ of type (F^*, R^*), i.e. σ_μ, for $\mu < \alpha$, is a continuous mapping of the space X into the space B. We say that an *α-system $\sigma = \{\sigma_\mu, \mu < \alpha\}$ fulfils formula* (ii) if the relation $q\big(\sigma_\mu(x_\xi), \mu < \alpha, \xi < \beta\big)$ holds in the model $\boldsymbol{B}$. Let $\mathfrak{B}$ be an arbitrary class of topological models of type (F^*, R^*) and let P be any set of formulas of the logic L^*. Any pair $(\sigma, \boldsymbol{B})$, where $\boldsymbol{B} \in \mathfrak{B}$ and σ is a topological α-system of mappings of X into $\boldsymbol{B}$ which fulfils every formula belonging to the set P, is said to be a *topological α-system of $\mathfrak{B}$-P-mappings of the space X*. The topological free α-system $(\sigma, \boldsymbol{C})$ of mappings of X in the class of all topological α systems of $\mathfrak{B}$-P-mappings of X is called the *topological $\mathfrak{B}$-free α-system of $\mathfrak{B}$-P-mappings of space X* and the model $\boldsymbol{C}$ is said to be the *topological α-free model determined by X and P in the class $\mathfrak{B}$*; P is called the *set of defining relation of $\boldsymbol{C}$* and the α-system σ is said to be the *canonical α-system of P-mappings of X into $\boldsymbol{C}$*. A topological $\mathfrak{B}$-free α-system of $\mathfrak{B}$-P-mappings of X, if it exists, is uniquely determined up to the relation $\equiv$, and a topological α-free model determined by X and P in the class $\mathfrak{B}$, if it exists, is uniquely determined up to topological strong isomorphisms.

Now we prove the next theorems

(7.8) *Let X be an arbitrary topological Hausdorff space and let $\mathfrak{B}$ be any quasi-primitive class of topological models of type (F^*, R^*). Moreover, let P be any set (possibly empty) of productable formulas of the logic $L^* = L(F^*, R^*, \Phi \times X)$. Then there exists a topological $\mathfrak{B}$-free α-system $(\sigma, \boldsymbol{C})$ of $\mathfrak{B}$-P-mappings of X. The model $\boldsymbol{C}$ is algebraically generated by the set $\sigma(X)$.*

(7.9) *Let X be an arbitrary topological Hausdorff space and let $\mathfrak{B}$ be any quasi-primitive class of topological models of type (F^*, R^*). Moreover, let P be any set (possibly empty) of productable formulas of the logic $L^* = L(F^*, R^*, \Phi\times X)$. Then there exists a topological a-free model $\boldsymbol{C}$ determined by X and P in the class $\mathfrak{B}$. The model $\boldsymbol{C}$ is algebraically generated by the set $\sigma(X)$, where σ is the canonical a-system of P-mappings of X into $\boldsymbol{C}$.*

Proof. The class A of all topological a-systems of $\mathfrak{B}$-P-mappings of X is quasi-primitive and closed with respect to relative topological subsystem, since $\mathfrak{B}$ is quasi-primitive and P is a set of productable formulas of logic L^*. Thus (7.8) follows from Theorem 1 and 2. Theorem (7.8) immediately implies (7.9) and the proof of (7.8) and (7.9) is thus finished.

Let $\boldsymbol{A}_t$, for $t\in T$, be any family of topological models of type (F^*, R^*), and let us assume that for every $t, u\in T$ the models $\boldsymbol{A}_t$ and $\boldsymbol{A}_u$ have the relative topological submodels $\boldsymbol{A}_{tu}$ and $\boldsymbol{A}_{ut}$ that are topologically strongly isomorphic. Moreover, let h_{tu} be a topological strong isomorphism of $\boldsymbol{A}_{tu}$ onto $\boldsymbol{A}_{ut}$. Let $\mathfrak{B}$ be an arbitrary quasi-primitive class of topological models of type (F^*, R^*). Let us denote by P the set of formulas of the logic $L^* = L(F^*, R^*, \Phi\times X)$, where $X = \mathop{\mathrm{S}}_{t\in T} A_t$ is the topological direct sum of A_t, having the form

$$\varphi_\mu(\langle t, a\rangle) = \varphi_\mu\big(\langle u, h_{tu}(a)\rangle\big),$$

where $\mu < a$, $t\in T$, $u\in T$ and $a\in A_{tu}$. Then, by (7.9), there exists a topological a-free model $\boldsymbol{C}$ determined by X and P in the class $\mathfrak{B}$. The model $\boldsymbol{C}$ is called the *topological a-$\mathfrak{B}$-free sum* (resp. *a-$\mathfrak{B}$-direct sum*) *of $\boldsymbol{A}_t$, $t\in T$, with identification of the submodels $\boldsymbol{A}_{tu}$ and $\boldsymbol{A}_{ut}$.*

If all the submodels $\boldsymbol{A}_{tu}$ are empty, then $\boldsymbol{C}$ is the *topological a-$\mathfrak{B}$-free sum* (resp. *a-$\mathfrak{B}$-direct sum*) *of models $\boldsymbol{A}_t$, $t\in T$* (3).

Theorems 1 and 2 in paper [2] of Malcev immediately result from (7.8) and (7.9). Indeed, putting $a = 1$ (i.e. $\Phi = \{\varphi_0\}$), $R^* = \emptyset$ in (7.8) and (7.9) and take $\mathfrak{B}$ as a primitive class of topological algebras of type F^*, we obtain from (7.8) and (7.9) the above mentioned theorems of Malcev.

§ 8. Remarks and problems.

A. Let $\boldsymbol{A}$ be an arbitrary topological model of type (F, R) and let $\mathfrak{B}$ be any quasi-primitive classes of topological models of type (F^*, R^*). Moreover, let P be any set (possibly empty) of productable basic map-

(3) This topological a-$\mathfrak{B}$-free sum (resp. a-$\mathfrak{B}$-direct sum) of $F_t, t\in T$, coincides with the topological a-$\mathfrak{B}$-direct sum from § 6 for $F = F^*$, $R = R^*$ and $P = \emptyset$.

ping-formulas of type $(F, R; F^*, R^*)$. By theorem (5.3) there exists a topological a-P-free model $\boldsymbol{C}$ determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$. Considering the model $\boldsymbol{A}$ and the models of the class $\mathfrak{B}$ without topology we obtain, by theorem (5.4), an algebraic model $\boldsymbol{C}'$ which is algebraic a-P-free determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$. Now consider the following question:

(S_1) *Are the models $\boldsymbol{C}$ and $\boldsymbol{C}'$ algebraically strongly isomorphic?*

Let $\sigma = \{\sigma_\mu, \mu < a\}$ and $\sigma' = \{\sigma'_\mu, \mu < a\}$ be canonical a-systems of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{C}$ and of $\boldsymbol{A}$ into $\boldsymbol{C}'$. Then we have $(\sigma', \boldsymbol{C}') \leqslant (\sigma, \boldsymbol{C})$ algebraically, i.e. there exists a unique algebraic homomorphism h of $\boldsymbol{C}'$ into $\boldsymbol{C}$ such that $\sigma = h\sigma'$. Since the sets $\sigma(A)$ and $\sigma'(A)$ generate algebraically the models $\boldsymbol{C}$ and $\boldsymbol{C}'$, h maps $\boldsymbol{C}'$ onto $\boldsymbol{C}$. If h is a strong isomorphism of $\boldsymbol{C}'$ onto $\boldsymbol{C}$, then obviously we have a positive answer to the question (S_1). Let us observe that

(8.1) *h is one-to-one, i.e. h is an isomorphism of $\boldsymbol{C}'$ onto $\boldsymbol{C}$ (possibly not strong) if and only if for any inequality of the logic $L^* = L(F^*, R^*, \Phi \times X)$*

$$(*) \qquad \tau\big(\varphi_\mu(x_\xi), \mu < a, \xi < \beta\big) \neq \vartheta\big(\varphi_\mu(x_\xi), \mu < a, \xi < \beta\big)$$

and for any sequence $(a_\xi, \xi < \beta) \in A^\beta$ if the relation

$$\tau\big(\sigma'_\mu(a_\xi), \mu < a, \xi < \beta\big) \neq \vartheta\big(\sigma'_\mu(a_\xi), \mu < a, \xi < \beta\big)$$

holds in the model $\boldsymbol{C}'$, then there exists a topological a-system $\sigma'' = \{\sigma''_\mu, \mu < a\}$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ into a model $\boldsymbol{B}$ such that the relation

$$\tau\big(\sigma''_\mu(a_\xi), \mu < a, \xi < \beta\big) \neq \vartheta\big(\sigma''_\mu(a_\xi), \mu < a, \xi < \beta\big)$$

holds in the model $\boldsymbol{B}$.

(8.2) *h is a strong isomorphism of $\boldsymbol{C}'$ onto $\boldsymbol{C}$ if and only if h is an isomorphism of $\boldsymbol{C}'$ onto $\boldsymbol{C}$ and for any term $\tau(\varphi_\mu(x_\xi), \mu < a, \xi < \beta)$ of the logic $L^* = L(F^*, R^*, \Phi \times X)$ and for any sequence $(a_\xi, \xi < \beta) \in A^\beta$ if there exists the element $\tau\big(\sigma_\mu(a_\xi), \mu < a, \xi < \beta\big)$ in $\boldsymbol{C}$, then there exists an element $\tau\big(\sigma'_\mu(a_\xi), \mu < a, \xi < \beta\big)$ in $\boldsymbol{C}'$.*

By (8.1) and (8.2) we have a sufficient condition for an affirmative answer to question (S_1). The problem of determining sufficient and necessary conditions for an affirmative answer to question (S_1) is open (**P 608**).

B. Let $\boldsymbol{A}$ be an arbitrary topological model of type (F, R) and let $\mathfrak{B}$ be any quasi-primitive class of topological models of type (F^*, R^*). Moreover, let P be any set (possibly empty) of productable basic mapping formulas of type $(F, R; F^*, R^*)$. Now let us consider the following question:

(S_2) *Does there exist a topological a-system of $\mathfrak{B}$-P-extensions of $\boldsymbol{A}$?*

Let the pair $(\sigma, \boldsymbol{C})$, where $\sigma = \{\sigma_\mu, \mu < a\}$, be a topological $\mathfrak{B}$-free a-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ (it exists by Theorem 4). By theorem (4.2) question (S_2) is equivalent to the question

(S_3) *Is the topological $\mathfrak{B}$-free a-system $(\sigma, \boldsymbol{C})$ of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ a topological a-system of $\mathfrak{B}$-P-extensions of $\boldsymbol{A}$, i.e. are all the mappings σ_μ, $\mu < a$, topological homeomorphisms of $\boldsymbol{A}$ into $\boldsymbol{C}$?*

Now we prove

(8.3) *If there exists a family $(\sigma_t, \boldsymbol{B}_t)$, where $t \in T$ and $\sigma_t = \{\sigma_{t\mu}, \mu < a\}$, of topological a-systems of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ having the property*

(U) *for each element $a \in A$ and every neighbourhood U of a there exists a finite subset $T' \subset T$ and there are neighbourhoods $U_{t\mu}$ of elements $\sigma_{t\mu}(a)$, where $t \in T'$ and $\mu < a$ such that*

$$\bigcap_{t \in T'} \sigma_{t\mu}^{-1}(U_{t\mu}) \subset U \quad \textit{for} \quad \mu < a,$$

then there exists a topological a-system of $\mathfrak{B}$-P-extensions of $\boldsymbol{A}$.

Proof. Let $(\sigma, \boldsymbol{B})$, where $\sigma = \{\sigma_\mu, \mu < a\}$ and $\boldsymbol{B} = \mathop{\mathrm{P}}\limits_{t \in T} \boldsymbol{B}_t$, be the topological direct product of a-systems $(\sigma_t, \boldsymbol{B}_t)$, $t \in T$. Obviously $(\sigma, \boldsymbol{B})$ is a topological a-system of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$. For all $\mu < a$, the mapping σ_μ is one-to-one. Indeed, let us assume that $\sigma_\mu(a) = \sigma_\mu(b)$ and $a \neq b$. Then we have

$$\sigma_{t\mu}(a) = \sigma_{t\mu}(b) \quad \text{for all } t \in T \text{ and } a \neq b.$$

Let U be a neighbourhood of a such that b does not belong to U. By condition (U) there are a finite subset $T' \subset T$ and neighbourhoods $U_{t\mu}$ of elements $\sigma_{t\mu}(a) = \sigma_{t\mu}(b)$, where $t \in T'$, such that

$$\bigcap_{t \in T'} \sigma_{t\mu}^{-1}(U_{t\mu}) \subset U,$$

and thus $b \in U$, which is impossible. Therefore σ_μ is one-to-one. Moreover, the mapping σ_μ^{-1}, for $\mu < a$, is continuous. Indeed, let us denote by U any neighbourhood of the element $a = \sigma_\mu^{-1}(\chi)$, where $\chi \in B$. By condition (U) there are a finite subset $T' \subset T$ and neighbourhoods $U_{t\mu}$ of elements $\sigma_{t\mu}(a)$, where $t \in T'$, such that

$$(**) \qquad \bigcap_{t \in T'} \sigma_{t\mu}^{-1}(U_{t\mu}) \subset U.$$

We define $V_\mu = \bigcap\limits_{t \in T'} p_t^{-1}(U_{t\mu})$, where p_t is the natural projection of B onto B_t. Then V_μ is a neighbourhood of χ and by $(**)$ we have

$\sigma_\mu^{-1}(V_\mu) \subset U$ and thus the mapping σ_μ^{-1} is continuous. Therefore the pair $(\sigma, \boldsymbol{B})$ is a topological α-system of $\mathfrak{B}$-P-extensions of $\boldsymbol{A}$ and the proof of (8.3) is finished.

Theorem (8.3) gives a sufficient condition for an affirmative answer to question (S_2) (resp. (S_3)).

The problem of determining sufficient and necessary conditions for an affirmative answer to question (S_2) (resp. (S_3)) is open (**P 609**).

C. Let $\boldsymbol{A}$ be any (topological) model of type (F, R) and let $\boldsymbol{B}$ be any (topological) model of type (F^*, R^*). For any α-system $\sigma = \{\sigma_\mu, \mu < \alpha\}$ of mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ we shall denote by h_σ the direct product of mappings σ_μ, $\mu < \alpha$. Then h_σ is the unique mapping of A into the direct power B^α such that $p_\mu h_\sigma = \sigma_\mu$ for all $\mu < \alpha$. A (topological) model $\boldsymbol{D}$ of type (F, R) is said to be a *(topological) P-product over* $\boldsymbol{B}$, where P is a set of basic mapping-formulas of type $(F, R; F^*, R^*)$ and $\boldsymbol{B}$ is a (topological) model of type (F^*, R^*), if it has the following properties:

1° $D = B^\alpha$.

2° For any (topological) model $\boldsymbol{A}$ of type (F, R) and for any (topological) α system σ of mappings of $\boldsymbol{A}$ into $\boldsymbol{B}$ the α-system σ is a (topological) α-system of P-mappings if and only if the mapping h_σ is a (topological) homomorphism of $\boldsymbol{A}$ into $\boldsymbol{D}$.

Let P be an arbitrary set of basic mapping-formulas of type $(F, R; F^*, R^*)$. Let us consider the following property of P:

(S_4) *Every (topological) model* $\boldsymbol{B}$ *of type* (F^*, R^*) *has a (topological) P-product over* $\boldsymbol{B}$.

The general problem of determining all sets P having the property (S_4) is open (**P 610**). Now we shall give some sufficient conditions for (S_4). Let us consider the following basic mapping-formulas of type $(F, R; F^*, R^*)$:

$$(\mathrm{i}_{f\varrho}) \qquad x = f(x_\xi, \xi < n(f)) \to \varphi_\varrho(x) = \tau_{f\varrho}(\varphi_\mu(x_\xi), \mu < \alpha, \xi < n(f)),$$

$$(\mathrm{i}_r) \qquad r(x_\xi, \xi < n(r)) \to q_r(\varphi_\mu(x_\xi), \mu < \alpha, \xi < n(r)),$$

where $f \in F$, $r \in R$, $\varrho < \alpha$, $\tau_{f\varrho}$ is a term of logic $L^* = L(F^*, R^*, \Phi \times X)$ and q_r is a formula of logic $L^* = L(F^*, R^*, \Phi \times X)$.

Now we shall prove

(8.4) *Every set P of all basic mapping-formulas of type* $(F, R; F^*, R^*)$ *having the forms* $(i_{f\varrho})$ *and* (i_r), *where* $f \in F$ *and* $r \in R$ *and* $\varrho < \alpha$, *has property* (S_4).

Proof. Let $\boldsymbol{B}$ be a model of type (F^*, R^*). The model $\boldsymbol{D} = \langle D, (f_{\boldsymbol{D}}) f \in F, (r_{\boldsymbol{D}}) r \in R\rangle$ of type (F, R) such that

1° $D = B^\alpha$,

2° $f_{\boldsymbol{D}}(\chi_\xi, \xi < n(f)) = \chi$ if and only if for all $\varrho < \alpha$ we have

$$\chi(\varrho) = \tau_{f\varrho}(\chi_\xi(\mu), \mu < \alpha, \xi < n(f)),$$

3° $r_{\boldsymbol{D}}(\chi_\xi, \xi < n(r))$ holds in $\boldsymbol{D}$ if and only if the relation $q_r(\chi_\xi(\mu), \mu < \alpha, \xi < n(r))$ holds in model $\boldsymbol{B}$,

is the P-product over $\boldsymbol{B}$.

The basic mapping-formulas of the form $(\mathrm{i}_{f\varrho})$ for algebras with finitary operations are considered in paper [1]. The basic mapping-formulas of the form $(\mathrm{i}_{f\varrho})$ for arbitrary quasi-algebras are considered in my papers [10, 8].

D. Let P be any set of basic mapping-formulas of type $(F, R; F^*, R^*)$. Let $\boldsymbol{A}$ be any model of type (F, R), and, moreover, let $\boldsymbol{B}$ be any model of type (F^*, R^*). A subset Y of A is said to be *P-$\boldsymbol{B}$-independent* if any α-system $\sigma = \{\sigma_\mu, \mu < \alpha\}$ of mappings of Y into $\boldsymbol{B}$ may be extended to an α-system $\sigma' = \{\sigma'_\mu, \mu < \alpha\}$ of P-mappings of $C_{\boldsymbol{A}}\, Y$ into $\boldsymbol{B}$. The P-independence for $R = R^* = \emptyset$ and for the set P having only formulas of the form $(\mathrm{i}_{f\varrho})$ is considered in my paper [10] and it has similar properties as the independence with respect to homomorphisms (see [3, 4]). We shall consider the notion of P-independence for arbitrary set P in the next paper.

E. The basic mapping-formulas of type $(F, R; F^*, R^*)$ obtained by using the set $\Phi = \{\varphi_\mu, \mu < \alpha\}$ are called *α-basic mapping-formulas of type* $(F, R; F^*, R^*)$. Let α and γ be any ordinal numbers with $\alpha < \gamma$. Then the α-basic mapping-formulas of type $(F, R; F^*, R^*)$ may be considered as some γ-basic mapping-formulas of type $(F, R; F^*, R^*)$. Let $\boldsymbol{A}$ be any topological model of type (F, R) and let $\mathfrak{B}$ be any quasi-primitive class of topological models of type (F^*, R^*). Moreover, let P be any set (possibly empty) of productable α-basic mapping-formulas of type $(F, R; F^*, R^*)$. We consider the set P also as a set of γ-basic mapping-formulas, where $\gamma > \alpha$. By Theorem 4 there are topological $\mathfrak{B}$-free α-system $(\sigma, \boldsymbol{C}_\alpha)$, where $\sigma = \{\sigma_\mu, \mu < \alpha\}$, of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$ and a topological $\mathfrak{B}$-free γ-system $(\sigma', \boldsymbol{C}_\gamma)$, where $\sigma' = \{\sigma'_\mu, \mu < \gamma\}$, of $\mathfrak{B}$-P-mappings of $\boldsymbol{A}$. Let us observe that an α-system $\sigma' | \alpha = \{\sigma'_\mu, \mu < \alpha\}$ is a topological α-system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{C}_\gamma$, and thus there exists only one topological homomorphism $h_{\alpha\gamma}$ of $\boldsymbol{C}_\alpha$ into $\boldsymbol{C}_\gamma$ such that $\sigma'_\mu = h_{\alpha\gamma}\sigma_\mu$ for $\mu < \alpha$. Moreover, every topological γ-system $\sigma'' = \{\sigma''_\mu, \mu < \gamma\}$ of mappings of $\boldsymbol{A}$ into $\boldsymbol{C}_\alpha$ such that $\sigma''_\mu = \sigma_\mu$ for $\mu < \alpha$ is a topological γ-system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{C}_\alpha$. Therefore there exists a topological homomorphism $h_{\gamma\alpha}$ (not unique) of $\boldsymbol{C}_\gamma$ into $\boldsymbol{C}_\alpha$ such that $\sigma_\mu = h_{\gamma\alpha}\sigma'_\mu$ for $\mu < \alpha$. Hence we have $\sigma_\mu = h_{\gamma\alpha}h_{\alpha\gamma}\sigma_\mu$ for all $\mu < \alpha$, i.e. $\sigma = h_{\gamma\alpha}h_{\alpha\gamma}\sigma$, and thus $h_{\gamma\alpha}h_{\alpha\gamma} = I$, where I is the identity mapping of $\boldsymbol{C}_\alpha$ onto $\boldsymbol{C}_\alpha$. Therefore $h_{\alpha\gamma}$ is one-to-one (possibly not onto) mapping of $\boldsymbol{C}_\alpha$ into $\boldsymbol{C}_\gamma$.

Moreover, the mapping $h_{\gamma a}^{-1}$ is continuous, since if $c = h_{a\gamma}^{-1}(d)$, then $c = h_{\gamma\varphi}(d)$; but $h_{\gamma a}$ is continuous. Thus we have proved the theorem

(8.4) *The mapping $h_{a\gamma}$ is a topological isomorphism (possibly not strong) of $\boldsymbol{C}_a$ into $\boldsymbol{C}_\gamma$ and the reverse mapping $h_{a\gamma}^{-1}$ is continuous.*

Since we have $\sigma'_\mu = h_{a\gamma}h_{\gamma a}\sigma'_\mu$ only for $\mu < a$, it may happen that $h_{a\gamma}h_{\gamma a} \neq I'$, where I' is the identity mapping of $\boldsymbol{C}_\gamma$ onto $\boldsymbol{C}_\gamma$. Now let us consider the following property of the pair $(\mathfrak{B}, P)$:

(S_5) *For ordinal numbers $\gamma > a$ and all topological models $\boldsymbol{A}$ of type (F, R) the a-P-free topological model $\boldsymbol{C}_a$ determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$ and the topological γ-P-free model $\boldsymbol{C}_\gamma$ determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$ are topologically strongly isomorphic.*

The general problem of determining all the pairs $(\mathfrak{B}, P)$ having property (S_5) is open (**P 611**).

We do not know if there exist pairs $(\mathfrak{B}, P)$ which satisfy (S_5).

Now we prove

(8.5) *Let $\mathfrak{B}$ be an arbitrary quasi-primitive class of topological models of type (F^*, R^*) having operations only, and let $\boldsymbol{A}$ be an arbitrary topological model of type (F, R). Moreover, let P be any set (possibly empty) of productable a-basic mapping-formulas of type $(F, R; F^*, R^*)$ and let $\boldsymbol{C}_a$ and $\boldsymbol{C}_\gamma$, where $\gamma > a$, be topological a-P-free and γ-P-free models determined by $\boldsymbol{A}$ in the class $\mathfrak{B}$. Then the mapping $h_{a\gamma}$ is a topological strong isomorphism of $\boldsymbol{C}_a$ onto the relative topological submodel of $\boldsymbol{C}_\gamma$ algebraically generated by the set $\bigcup_{\mu<a} \sigma'_\mu(A)$, where $\sigma' = \{\sigma'_\mu, \mu < \gamma\}$, is the canonical γ-system of P-mappings of $\boldsymbol{A}$ onto $\boldsymbol{C}_\gamma$.*

Proof. Theorem (8.5) immediately follows from (8.4) and from the proof of (8.4). Moreover, let us observe that $(\sigma, \boldsymbol{C}_a) \equiv (\sigma' | a, \boldsymbol{C}')$, where $\boldsymbol{C}'$ is the relative submodel of $\boldsymbol{C}_\gamma$ algebraically generated by $\bigcup_{\mu<a} \sigma'_\mu(A)$, $\sigma' | a = \{\sigma'_\mu, \mu < a\}$ and σ is the canonical a-system of P-mappings of $\boldsymbol{A}$ into $\boldsymbol{C}_a$, since we have $\sigma'_\mu = h_{a\gamma}\sigma_\mu$ for $\mu < a$, where $\sigma = \{\sigma_\mu, \mu < a\}$.

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