COLLOQUIUM MATHEMATICUM

VOL. XXXII 1975 FASC. 2

# ON THE TWO DEFINITIONS OF INDEPENDENCE

BY

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**1. Introduction.** Let $(X, \boldsymbol{A}, P)$ be a probability space. Two sub-$\sigma$-algebras $\boldsymbol{B}$ and $\boldsymbol{C}$ of $\boldsymbol{A}$ are said to be *independent* if $P(B \cap C) = P(B)P(C)$ whenever $B \in \boldsymbol{B}$ and $C \in \boldsymbol{C}$. For any real-valued $\boldsymbol{A}$-measurable function $f$ defined on $X$, put

$$\boldsymbol{B}_f = \{f^{-1}B\colon B \text{ is a linear Borel set}\}$$

and

$$\boldsymbol{A}_f = \{f^{-1}C \in \boldsymbol{A}\colon C \text{ is a linear set}\}.$$

Two real-valued $\boldsymbol{A}$-measurable functions $f$ and $g$ are said to be

(1) *independent according to the Steinhaus definition* if $\boldsymbol{B}_f$ and $\boldsymbol{B}_g$ are independent, and

(2) *independent according to the Kolmogorov definition* if $\boldsymbol{A}_f$ and $\boldsymbol{A}_g$ are independent.

In general, the two definitions are not equivalent (see [2] and [4]).

Gnedenko and Kolmogorov [3] introduced the concept of perfectness of a probability measure. A probability measure $P$ on $(X, \boldsymbol{A})$ is called *perfect* if, for every real-valued $\boldsymbol{A}$-measurable $f$ on $X$ and for every subset $A$ of the real line for which $f^{-1}A \in \boldsymbol{A}$, there is a linear Borel set $B \subset A$ such that $P(f^{-1}B) = P(f^{-1}A)$. Doob (see appendix in [3]) has noted that the two definitions of independence are equivalent if $P$ is perfect.

The requirement of perfectness of probability measures is sufficient to refine Kolmogorov's model for probability theory, so that it is technically pleasing (see appendix in [3] and introduction in [1]). But the necessity of perfectness for a technically pleasing model has not been investigated so far. Rodine [7] raised the following question in this direction:

If the two definitions of independence are equivalent in a probability space $(X, \boldsymbol{A}, P)$, is then $P$ perfect?

We show, using the notion of a strongly Blackwell space, that the answer is in the negative. We also study some related problems.

All measures considered in this paper are probabilities. A $\sigma$-algebra $\boldsymbol{C}$ of subsets of a set $Z$ is said to be *separable* if it is countably generated and contains all singletons.

**2. Main results.** Let $(X, \boldsymbol{A})$ be a Borel space, where $\boldsymbol{A}$ is separable. $(X, \boldsymbol{A})$ is said to be a *Blackwell space* if, for any separable sub-$\sigma$-algebra $\boldsymbol{A}_1$ of $\boldsymbol{A}$, we have $\boldsymbol{A}_1 = \boldsymbol{A}$. A space $(X, \boldsymbol{A})$ is said to be *strongly Blackwell* if any two countably generated sub-$\sigma$-algebras of $\boldsymbol{A}$ with the same atoms are identical. Strongly Blackwell spaces have been independently introduced by Ashok Maitra (oral communication) and Ryll-Nardzewski [8].

THEOREM. *For a Borel space $(X, \boldsymbol{A})$, where the $\sigma$-algebra $\boldsymbol{A}$ is separable, the following conditions are equivalent:*

(i) *$(X, \boldsymbol{A})$ is a strongly Blackwell space.*

(ii) *If $\boldsymbol{B}$ is a countably generated sub-$\sigma$-algebra of $\boldsymbol{A}$ and $A \in \boldsymbol{A}$ is a union of $\boldsymbol{B}$-atoms, then $A \in \boldsymbol{B}$.*

(iii) *For every $\boldsymbol{A}$-measurable real-valued function $f$ defined on $X$, we have $\boldsymbol{B}_f = \boldsymbol{A}_f$.*

Proof. (i) $\Rightarrow$ (ii). Indeed, if $A \in \boldsymbol{A}$ is a union of $\boldsymbol{B}$-atoms, then, by (i), $\sigma\{\boldsymbol{B}, A\} = \boldsymbol{B}$.

(ii) $\Rightarrow$ (iii). The sub-$\sigma$-algebra $\boldsymbol{B}_f = \{f^{-1}B\colon B$ is a linear Borel set$\}$ of $\boldsymbol{A}$ is countably generated. Moreover, every set in $\boldsymbol{A}_f$ is a union of $\boldsymbol{B}_f$-atoms. Hence, by (ii), $\boldsymbol{A}_f \subset \boldsymbol{B}_f$.

(iii) $\Rightarrow$ (i). Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be two countably generated sub-$\sigma$-algebras of $\boldsymbol{A}$ with the same atoms. Let $\boldsymbol{A}_1 = \sigma\{A_n\}$ and let $f$ be the Marczewski function of $\{A_n\}$, i.e,

$$f = \sum_{n=1}^{\infty} (2/3^n) 1_{A_n}.$$

It is easy to check that $f$ is $\boldsymbol{A}$-measurable and that $\boldsymbol{B}_f = \boldsymbol{A}_1$. Now, since $\boldsymbol{A}_2 \subset \boldsymbol{A}_f$, (iii) implies $\boldsymbol{A}_2 \subset \boldsymbol{A}_1$. Similarly, $\boldsymbol{A}_1 \subset \boldsymbol{A}_2$, so that (i) holds.

COROLLARY 1. *Let $(X, \boldsymbol{A})$ be a strongly Blackwell space and let $P$ be any measure on $(X, \boldsymbol{A})$. Then the two definitions of independence are equivalent in $(X, \boldsymbol{A}, P)$.*

Proof. For every real-valued $\boldsymbol{A}$-measurable function $f$ defined on $X$, by the theorem, $\boldsymbol{B}_f = \boldsymbol{A}_f$. Hence, whatever be the measure $P$ on $(X, \boldsymbol{A})$, the two definitions of independence are equivalent.

Now we present an example to show that the answer to the question raised in section 1 is in the negative.

Example 1. Ryll-Nardzewski [8] has shown the existence of a non-Lebesgue measurable subset $X^*$ of the unit interval $[0, 1]$, such that the Borel space $(X^*, \boldsymbol{A}^*)$, where

$$\boldsymbol{A}^* = \{B \cap X^*\colon B \text{ is a linear Borel set}\},$$

is strongly Blackwell. Further, $X^*$ is thick, that is, $X^*$ has outer Lebesgue measure one. Let $P^*$ be the trace of outer Lebesgue measure on $(X^*, \mathbf{A}^*)$. Then $(X^*, \mathbf{A}^*, P^*)$ is a non-perfect probability space. But, in view of corollary 1, the two definitions of independence are equivalent in $(X^*, \mathbf{A}^*, P^*)$.

Remark 1. The following question is a global version of the question raised in section 1:

Suppose that $(X, \mathbf{A})$ is a Borel space such that, whatever be the probability measure $P$ on $(X, \mathbf{A})$ we consider, the two definitions of independence are equivalent in $(X, \mathbf{A}, P)$. Then does there exist a non-atomic perfect measure on $(X, \mathbf{A})$?

Here we demand something weaker after having put a stronger assumption on $(X, \mathbf{A})$. Still the answer to this question is in the negative as we show below.

Suppose $X$ is a subset of the real line. Let

$$\mathbf{B}_X = \{B \cap X\colon B \text{ is a linear Borel set}\}.$$

Then we have

P1 ([9], Lemma 3). *In order that every measure on $(X, \mathbf{B}_X)$ be perfect it is necessary and sufficient that the set $X$ be universally measurable.*

A subset $D$ of the real line is said to be a *perfect set* if $D = \{$limit points of $D\}$. Every uncountable Borel subset of the real line contains a perfect set (see [5], p. 447). The following result, which is easy to prove, is in contrast with P1:

P2. *Every non-atomic measure on $(X, \mathbf{B}_X)$ is non-perfect if and only if $X$ does not contain any perfect set.*

Example 2. The set $X^*$, constructed by Ryll-Nardzewski (example 1), can be taken so that both $X^*$ and $[0, 1] \setminus X^*$ do not contain any perfect set. Hence $(X^*, \mathbf{A}^*)$, defined as in example 1, is a Borel space such that the two definitions of independence are equivalent in $(X^*, \mathbf{A}^*, P^*)$, no matter what measure $P^*$ we consider, yet every non-atomic measure on $(X^*, \mathbf{A}^*)$ is non-perfect by P2.

Remark 2. We also note that if $(X^*, \mathbf{A}^*)$ is a strongly Blackwell space, then, for every sub-$\sigma$-algebra $\mathbf{B}$ of $\mathbf{A}^*$ and for every measure $P$ on $(X^*, \mathbf{B})$, the two definitions of independence are equivalent in $(X^*, \mathbf{B}, P)$.

We can raise the following question:

Is $(X, \mathbf{A})$ strongly Blackwell if $(X, \mathbf{A})$ is a Borel space such that

(a) $\mathbf{A}$ is separable, and

(b) for every probability measure $P$ on $(X, \mathbf{A})$, the two definitions of independence are equivalent in $(X, \mathbf{A}, P)$?

The answer to this question is also in the negative as shown by the following example:

Example 3. Let $X_1$ be a coanalytic subset of $[0,1]$, such that if $\boldsymbol{A}_1 = \{B \cap X_1: B$ is a linear Borel set$\}$, then $(X_1, \boldsymbol{A}_1)$ is not a Blackwell space. Such a coanalytic set $X_1$ has been constructed by Maitra [6]. Now, by P1, every measure on $(X_1, \boldsymbol{A}_1)$ is perfect. Hence $(X_1, \boldsymbol{A}_1)$ is a Borel space satisfying conditions (a) and (b) of the question (cf. also section 1). But $(X_1, \boldsymbol{A}_1)$ fails to be a strongly Blackwell space, since it is not even a Blackwell space.

Next we modify example 1 so as to get a non-perfect probability space in which the two definitions of independence are equivalent, but whose underlying Borel space is not Blackwell.

Example 4. Let $(X^*, \boldsymbol{A}^*, P^*)$ be as in example 1. Let $N$ be a subset of the interval $[2,3]$, such that the Borel space $(N, \boldsymbol{N})$, where

$$\boldsymbol{N} = \{B \cap N: B \text{ is a linear Borel set}\},$$

is not Blackwell (for instance, we can take $N = \{x+2: x \in X_1)$, where $X_1$ is as in example 3). Let

$$X = X^* \cup N \quad \text{and} \quad \boldsymbol{A} = \{A^* \cup N_1: A^* \in \boldsymbol{A}^*, N_1 \in \boldsymbol{N}\},$$

and let $P(A) = P^*(A \cap X^*)$ for $A \in \boldsymbol{A}$. It is easy to verify that $(X, \boldsymbol{A})$ is not a Blackwell space and that $P$ is non-perfect on $(X, \boldsymbol{A})$. We shall show that the two definitions of independence are equivalent in $(X, \boldsymbol{A}, P)$.

For every real-valued $\boldsymbol{A}$-measurable function $f$ defined on $X$, denote by $f^*$ the restriction of $f$ to $X^*$. Then we have $(\boldsymbol{A}_f \cap X^*) \subset \boldsymbol{A}^*_{f^*}$. Moreover, by the theorem, $\boldsymbol{A}^*_{f^*} = \boldsymbol{B}^*_{f^*}$. Hence $\boldsymbol{A}_f \cap X^* = \boldsymbol{B}_f \cap X^*$. Now, if $f$ and $g$ are two real-valued $\boldsymbol{A}$-measurable functions defined on $X$, and if $\boldsymbol{B}_f$ and $\boldsymbol{B}_g$ are independent, then $\boldsymbol{B}_f \cap X^*$ and $\boldsymbol{B}_g \cap X^*$ are independent. Hence $\boldsymbol{A}_f \cap X^*$ and $\boldsymbol{A}_g \cap X^*$ are independent and, consequently, so are $\boldsymbol{A}_f$ and $\boldsymbol{A}_g$. Thus the two definitions of independence are equivalent in $(X, \boldsymbol{A}, P)$.

**3. Comments.** It is not known whether, in every probability space whose underlying Borel space is Blackwell, the two definitions of independence are equivalent. (**P 930**)

In our example 4, if we remove the $P$-null set $N$ from $X$, then the resulting space is strongly Blackwell. We do not know whether there is a non-perfect probability space in which the two definitions of independence are equivalent and such that the underlying Borel space is not Blackwell even if we remove any null set. (**P 931**)

Finally, our examples point out that a nicer model for probability theory cannot be achieved by just demanding the equivalence of the two definitions of independence. Further, other pathologies like the non-existence of regular conditional probabilities also need to be avoided. It will be worth-while to investigate whether a condition less restrictive than perfectness exists which allows us to avoid these pathologies.

The author wishes to thank Professor Mahendra G. Nadkarni and Dr. N. D. Prabhakar for some useful discussions he had with them. He is also thankful to Dr. B. V. Rao and Dr. K. P. S. Bhaskara Rao for suggesting improvements in the original version of the paper.

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*Reçu par la Rédaction le 23. 7. 1973 ;*
*en version modifiée le 15. 3. 1974*