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APPROXIMATION OF SET-VALUED FUNCTIONS BY CONTINUOUS FUNCTIONS

BY

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Introduction. Let X be a topological space, E a Banach space. Let $C(X, E)$ be the space of continuous functions $f: X \to E$. Suppose F is a map from X into subsets of E. Define the distance of an $f \in C(X, E)$ from F by the relation

$$\varrho(f, F) = \sup_{x \in X} \sup_{y \in F(x)} \|f(x) - y\| \tag{0.1}$$

where $\| \ \|$ stands for the norm in E.

The main result of this note is concerned with the existence of the best approximation of a set-valued function F by a continuous point-valued function. That is we give conditions (cf. Theorem 1, Section 2) under which there exists an $f_0 \in C(X, E)$ such that

$$\varrho(f_0, F) = \inf_{f \in C(X,E)} \varrho(f, F). \tag{0.2}$$

In Section 3, we apply this result to answer the following question posed by Pełczyński [5].

Let X, Y be compact topological spaces and let $\varphi: Y \to X$ be a continuous surjection. By $\varphi^0: C(X, E) \to C(Y, E)$ we denote the conjugate map given by $\varphi^0 f = f \circ \varphi$ if $f \in C(X, E)$ (E as above is a Banach space.)

Question. For an arbitrary but fixed $h \in C(Y, E)$, let

$$d\big(h, \varphi^0 C(X, E)\big) = \inf_{g \in \varphi^0 C(X,E)} \|h - g\|, \tag{0.3}$$

where $\|h - g\| = \max_{y \in Y} \|h(y) - g(y)\|$. Does there exist a $g_0 \in \varphi^0 C(X, E)$ such that

$$d\big(h, \varphi^0 C(X, E)\big) = \|h - g_0\| \, ? \tag{0.4}$$

The answer to this question is affirmative if E is uniformly convex and is given by Theorem 2.

The last section concerns again the existence of the best approximation of a set-valued function F by continuous functions but (0.1) is replaced there by an essential supremum-type distance. Theorem 3 of Section 4 contains as a special case a recent result due to Holmes and Kripke [2] concerning approximations of real-valued bounded functions by continuous functions.

The proof of our main result strongly depends upon a theorem of E. Michael on the existence of continuous selections. This theorem along with some basic definitions is provided, for the convenience of the reader, in Section 1.

It is a pleasant duty for the author to thank Professor Z. Semadeni for calling the author's attention to Pełczyński's problem and Professor A. Pełczyński for a stimulating discussion and, in particular, for supplying a list of references connected with his problem.

1. Notation and definitions. Throughout this note X, Y will denote topological spaces, E a uniformly convex Banach space. Let us recall that a Banach space E is uniformly convex (Clarkson [1]) if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x-y\| \geqslant \varepsilon (x, y \in E)$, then $\|(x+y)/2\| \leqslant 1-\delta$. Without any loss of generality we may assume that $\delta(\varepsilon)$ is non-decreasing and, manifestly, that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By 2^E, $K(E)$ and $C(E)$ we denote the set of all subsets of E, closed subsets of E and closed convex subsets of E, respectively.

Let F be a map of X into 2^E. The map F is *upper semicontinuous* (u.s.c.) if the set $\{x \mid F(x) \subset G\}$ is open in X for each open $G \subset E$. Similarly, F is *lower semi-continuous* (l.s.c.) if the set $\{x \mid F(x) \cap G \neq \emptyset\}$ is open in X for each open $G \subset E$.

Put

$$r(x, F) = \inf_{y \in E} \sup_{z \in F(x)} \|y-z\| \tag{1.1}$$

and

$$r(F) = \sup_{x \in X} r(x, F). \tag{1.2}$$

For an arbitrary $F, r(x, F)$ for some x or $r(F)$ may be infinite. Since for each $f \in C(X, E)$ we have the inequality

$$\sup_{y \in F(x)} \|f(x)-y\| \geqslant r(x, F) \quad \text{for each } x \in X,$$

therefore by (0.1) and (1.2) we get

$$\varrho(f, F) \geqslant r(F) \quad \text{for each } f \in C(X, E). \tag{1.3}$$

Hence, also,

$$\varrho(F) = \inf_{f \in C(X,E)} \varrho(f, F) \geqslant r(F). \tag{1.4}$$

By $B(x, r), x \in E, r \geqslant 0$, we denote the open ball centered at x of radius r, and by $\bar{B}(x, r)$ the closed ball.

The following two propositions describe properties of uniformly convex Banach spaces we will need later. Proposition 1 is a slightly changed lemma given in [1], p. 3, but the proof of it, which we include here for convenience of the reader, is almost the same word for word.

PROPOSITION 1. *Let E be uniformly convex. If $\|x_1 - x_2\| \geqslant \varepsilon$, $x_1, x_2 \in E$, then, for any $r > 0$,*

$$(1.5) \qquad B\left(\frac{1}{2}(x_1+x_2), \big(1-\delta(\varepsilon/r)\big)r\right) \supset B(x_1, r) \cap B(x_2, r).$$

Proof. Let y belong to the right-hand side of (1.5). Without any loss of generality we may assume that $y = 0$. Therefore, to prove (1.5), we have to show that

$$(1.6) \qquad \|(x_1+x_2)/2\| \leqslant \big(1-\delta(\varepsilon/r)\big)r,$$

if $\|x_1\| \leqslant r$, $\|x_2\| \leqslant r$ and $\|x_1 - x_2\| \geqslant \varepsilon$. It is easy to see, by a proper dilation or contraction, that to prove (1.6) it's enough to show that

$$(1.7) \qquad \|(x_1+x_2)/2\| \leqslant 1-\delta(\varepsilon) \text{ if } \|x_1\| = 1, \|x_2\| \leqslant 1, \|x_1-x_2\| \geqslant \varepsilon.$$

There exist y_1, y_2 on the unit sphere such that $x_2 = \lambda_1 y_1 + \lambda_2 y_2$, where $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$ and $\|x_1 - y_1\| \geqslant \varepsilon$, $\|x_1 - y_2\| \geqslant \varepsilon$.

That such y_1, y_2 exist, follows from the existence of a supporting hyperplane to the ball $\bar{B}(x_1, \|x_1 - x_2\|)$ passing through x_2. By definition of uniform convexity we have

$$\begin{aligned} \|x_1+x_2)/2\| &\leqslant \lambda_1\|(x_1+y_1)/2\| + \lambda_2\|(x_1+y_2)/2\| \\ &\leqslant \lambda_1(1-\delta) + \lambda_2(1-\delta) = 1-\delta, \end{aligned}$$

which completes the proof.

PROPOSITION 2. *If E is uniformly convex, $r > 0$, $x, y \in E$ fixed, then there exists a function $\eta(\varepsilon) > 0$ defined and non-decreasing for $\varepsilon > 0$ and tending to 0 as $\varepsilon \to 0$ such that*

$$(1.8) \qquad \bar{B}(x, r) \cap \bar{B}(y, r+\varepsilon) \subset \bar{B}\big(z(\eta(\varepsilon)), r\big),$$

where $z(\eta) = y + \eta(x-y)/\|x-y\|$.

Proof. Put

$$(1.9) \qquad \eta(\varepsilon) = \inf\{\eta \mid \bar{B}(x, r) \cap \bar{B}(y, r+\varepsilon) \subset \bar{B}(z(\eta), r)\}.$$

Since $\eta = \|x-y\|$ belongs to the set in the right-hand side of (1.9), $\eta(\varepsilon)$ is well defined. It is easy to see that "inf" in (1.9) can be replaced by "min". Therefore to prove Proposition 2 it is enough to show that $\eta(\varepsilon)$ defined by (1.9) tends to zero as $\varepsilon \to 0$. Manifestly, by (1.9), $\eta(\varepsilon)$ is non-decreasing.

Suppose that $\lim\limits_{\varepsilon\to 0} \eta(\varepsilon) = \eta_0 > 0$. Let $\varepsilon_0 > 0$ be such that

$$0 < \eta_0 \leqslant \eta(\varepsilon) < 3\eta_0/2 \quad \text{if} \quad \varepsilon < \varepsilon_0. \tag{1.10}$$

By (1.9), Propositions 1, 2 and (1.10) we have for each $\varepsilon < \varepsilon_0$

$$\begin{aligned}\bar{B}(x, r) \cap \bar{B}(y, r+\varepsilon) &\subset \bar{B}\big(z(\eta(\varepsilon)), r+\varepsilon\big) \cap \bar{B}(y, r+\varepsilon)\\ &\subset \bar{B}\Big(\tfrac{1}{2}\big(z(\eta(\varepsilon))+y\big), \big(1-\delta(\eta_0/r+\varepsilon_0)\big)(r+\varepsilon)\Big).\end{aligned}$$

Choose $\varepsilon_1 > 0$ such that $\big(1-\delta(\eta_0/r+\varepsilon_0)\big)(r+\varepsilon_1) \leqslant r$, and note that $\big(z(\eta(\varepsilon))+y\big)/2 = z\big(\eta(\varepsilon)/2\big)$. This and the last inclusion prove that $\eta(\varepsilon_1)/2$ belongs to the set in the right-hand side of (1.9). But $\eta(\varepsilon_1) > 0$, thus a contradiction with (1.9), and hence $\eta_0 = 0$, which was to be proved.

Finally, let us state a theorem due to Michael [3] to be used in the next section. Before, let us recall that $F: X \to 2^E$ admits a (continuous) selection if there is an $f \in C(X, E)$ such that $f(x) \in F(x)$ for each $x \in X$.

THEOREM OF MICHAEL [3]. *The following properties of T_1-spaces are equivalent*:

(a) *X is paracompact.*

(b) *If E is a Banach space, then every l.s.c. F of X into $C(E)$ admits a selection.*

2. The main result. We will now prove the following

THEOREM 1. *Suppose that X is paracompact and E is uniformly convex Banach space. For each u.s.c. map $F: X \to K(E)$ there exists a best approximation by functions from $C(X, E)$; that is, there exists an $f_0 \in C(X, E)$ such that*

$$\varrho(f_0, F) = \inf_{f \in C(X,E)} \varrho(f, F).$$

Moreover, for each such f_0 we have the equality $\varrho(f_0, F) = r(F)$.

Proof. Note that if $r(F) = \infty$, then by (1.3) and (1.4) $\varrho(f, F) = +\infty$ for each $f \in C(X, E)$ and the Theorem is trivial. Thus the only interesting case is if $0 < r(F) < +\infty$.

Define

$$H(x) = \big\{p \in E \mid F(x) \subset \bar{B}\big(p, r(F)\big)\big\}, \quad x \in X. \tag{2.1}$$

We shall prove first that $H(x)$ is not empty closed and convex for each $x \in X$ and that the map $H: X \to C(E)$ is l.s.c. The closedness of $H(x)$

follows from closedness of $F(x)$ and (2.1). Suppose that $y_1, y_2 \in H(x)$, $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1+\lambda_2 = 1$. Then by (2.1) we have

$$\|y-\lambda_1 y_1-\lambda_2 y_2\| \leqslant \lambda_1\|y-y_1\|+\lambda_2\|y-y_2\| \leqslant r(F) \quad \text{for each } y \in F(x),$$

thus $H(x)$ is convex. If $r(x, F) < r(F)$, then clearly $H(x) \neq \emptyset$. Suppose then that $r(x_0, F) = r(F)$ for an $x_0 \in X$. Put

$$(2.2) \qquad H_\gamma = \{p \mid F(x_0) \subset \bar{B}(p, r(F)+\gamma)\}, \qquad \gamma > 0.$$

The set H_γ is not empty for each $\gamma > 0$. Let $\varepsilon > 0$ be arbitrary and choose γ such that

$$(2.3) \qquad r_1 = \Big(1-\delta\big(\varepsilon/(r(F)+\gamma)\big)\Big)\big(r(F)+\gamma\big) < r(F).$$

Since $\delta\big(\varepsilon/r(F)+\gamma\big) \geqslant \delta\big(\varepsilon/r(F)+1\big)$ if $\gamma \leqslant 1$, there is a γ with $0 < \gamma < 1$ satisfying (2.3). If (2.3) holds true and $p_1, p_2 \in H_\gamma$, then $\|p_1-p_2\| < \varepsilon$. Indeed, suppose the contrary. Then by Proposition 1, (2.2) and (2.3) we have $F(x_0) \subset \bar{B}\big((p_1+p_2)/2, r_1\big)$. Thus $r(x_0, F) \leqslant r_1 < r(F)$, which contradicts the assumption that $r(x_0, F) = r(F)$. We have proved that the diameter of H_γ is small if γ is small. Since $H_\gamma \subset H_\delta$ if $\gamma < \delta$, the intersection $\bigcap_{\gamma>0} H_\gamma$ is not empty and reduces to a single point p_0. It is obvious that $F(x_0) \subset \bar{B}\big(p_0, r(F)\big)$ and that $H(x_0) = \{p_0\}$. Thus $H(x)$ is not empty for each $x \in X$.

To prove that $H: X \to C(E)$ is l.s.c. consider the set

$$(2.4) \qquad A = \{x \in X \mid H(x) \cap G \neq \emptyset\},$$

where $G \subset E$ is a fixed open set. Let $x_0 \in A$ and $p_0 \in H(x_0) \cap G$. Since F is u.s.c., there exists, for each $\varepsilon > 0$, a neighborhood $N(\varepsilon)$ of x_0 such that

$$(2.5) \qquad F(x) \subset B\big(p_0, r(F)+\varepsilon\big) \quad \text{if} \quad x \in N.$$

Let ε be such that $\eta(\varepsilon)$ of Proposition 2 is smaller than η_0, where $B(p_0, \eta_0) \subset G$. By Proposition 2 and (2.5) we have

$$(2.6) \qquad F(x) \subset \bar{B}\big(p_1, r(F)\big) \cap \bar{B}\big(p_0, r(F)+\varepsilon\big) \subset \bar{B}\Big(z\big(\eta(\varepsilon)\big), r(F)\Big),$$

where $p_1 \in H(x)$ and $z\big(\eta(\varepsilon)\big) \in B(p_0, \eta_0)$. By (2.6), $z\big(\eta(\varepsilon)\big) \in H(x)$, too, whence $B(p_0, \eta_0) \cap H(x) \subset G \cap H(x) \neq \emptyset$. Since x is an arbitrary point of N, this shows that $N \subset A$. Hence A is open and H is l.s.c.

We can now apply Michael's Theorem, by which there is an $f_0 \in C(X, E)$ such that

$$f_0(x) \in H(x) \quad \text{for each} \quad x \in X.$$

This, (2.1) and (0.1) imply that $\varrho(f_0, F) \leqslant r(F)$, which in turn together with (1.3) shows that $\varrho(f_0, F) = r(F)$. This and (1.4) proves that f_0 is the best approximation as well as that

$$r(F) = \inf_{f \in C(X,E)} \varrho(f, F).$$

This completes the proof of Theorem 1.

3. An application. We will now apply Theorem 1 to answer Pełczyński's question stated in the introduction. In this section, X and Y are compact, $\varphi: Y \to X$ is a continuous surjection. By $\varphi^0: C(X, E) \to C(Y, E)$ we denote the conjugate map given by $\varphi^0 f = f \circ \varphi$ if $f \in C(X, E)$ (E, as above, is a uniformly convex Banach space).

Note that each $g \in \varphi^0 C(X, E)$ is constant on $\varphi^{-1}(x) = \{y \in Y \mid \varphi(y) = x\}$ for every $x \in X$. We have for each $h \in C(Y, E)$ the inequality

$$s(h) = \sup_{x \in X} \inf_{z \in E} \sup_{y \in \varphi^{-1}(x)} \|h(y) - z\| \leqslant d(h, \varphi^0(X, E)), \tag{3.1}$$

where d is given by (0.3). Indeed,

$$\|h - g\| = \sup_{x \in X} \sup_{y \in \varphi^{-1}(x)} \|h(y) - g(y)\| \geqslant \sup_{x \in X} \inf_{z \in E} \sup_{y \in \varphi^{-1}(x)} \|h(y) - z\| = s(h),$$

thus (3.1) follows from (0.3).

THEOREM 2. *For each $h \in C(Y, E)$ there exists the best approximation g_h of h by functions from $\varphi^0 C(X, E)$; that is, $g_h \in \varphi^0 C(X, E)$ and is such that $\|h - g_h\| = d(h, \varphi^0 C(X, E))$. Moreover, each such g_h satisfies the equality $\|h - g_h\| = s(h)$.*

In the case that E is the real line, the second part of Theorem 2 was given by Pełczyński [5] and a proof of the first part due to S. Mazur can be found in [7], p. 20 (cf. also [2]). In the case that E is the complex plane, the second part of Theorem 2 was obtained by Pełczyński [6].

Proof of Theorem 2. Because of (3.1) it is enough to prove the existence of a $g \in \varphi^0 C(X, E)$ such that

$$\|h - g\| = s(h). \tag{3.2}$$

Put

$$F(x) = \{z \in E \mid z = h(y),\ y \in \varphi^{-1}(x)\}. \tag{3.3}$$

Since φ is continuous and Y is compact, $\varphi^{-1}(x)$ is also compact for each $x \in X$, and so is $F(x)$. Hence (3.3) defines a map $F: X \to K(E)$. Suppose now that $f_0 \in C(X, E)$ is such that

$$\varrho(f_0, F) = r(F), \tag{3.4}$$

where ϱ and r are given by (0.1) and (1.2), respectively. Put $g_0 = \varphi^0 f_0 = f_0 \circ \varphi$. Then by (0.1) and (3.1) we get

$$\varrho(f_0, F) = \sup_{x \in X} \max_{z \in F(x)} \|z - f_0(x)\| = \sup_{x \in X} \max_{y \in \varphi^{-1}(x)} \|h(y) - g_0(y)\| = \|h - g_0\|.$$

On the other hand, by (1.1) and (3.1) we have

$$r(F) = \sup_{x \in X} \inf_{z \in E} \max_{y \in F(x)} \|y - z\| = \sup_{x \in X} \inf_{z \in E} \max_{y \in \varphi^{-1}(x)} \|h(y) - z\| = s(h).$$

Thus we see that if $f_0 \in C(X, E)$ satisfies (3.4), then $g_0 = f_0 \circ \varphi$ satisfies (3.2). Hence to complete the proof it is enough to check, because of Theorem 1, that F defined by (3.3) is u.s.c. To prove this let us take an open subset $G \subset E$. By (3.3), $F(x) \subset G$ if and only if $\varphi^{-1}(x) \subset h^{-1}(G)$. Since h is continuous, $h^{-1}(G)$ is an open subset of Y. Now, it is easy to check that

$$A = \{x \in X \mid \varphi^{-1}(x) \subset h^{-1}(G)\} = X \setminus \varphi\big(Y \setminus h^{-1}(G)\big). \tag{3.5}$$

Since $h^{-1}(G)$ is open and Y is compact, $Y \setminus h^{-1}(G)$ is also compact and so is $\varphi\big(Y \setminus h^{-1}(G)\big)$, because φ is continuous. Hence the set A given by (3.5) is open. But $A = \{x \in X \mid F(x) \subset G\}$. Therefore F is u.s.c. and Theorem 1 completes the proof of Theorem 2.

4. Approximation of bounded functions. In this section, E is a Euclidean space, X is paracompact. Let μ be a measure defined for all open subsets of X and such that $\mu(U) > 0$ for each open $U \subset X$. We denote by $\mathcal{N}$ the family of all μ-null subsets of X. Consider a map $F: X \to 2^E$. We say that F is *locally μ-essentially bounded* if for each $x \in X$ there is an open set $U \subset X$ and a μ-null set N such that $x \in U$ and $F_{|U \setminus N}$ is bounded ($F(x)$ is contained in a ball for each $x \in U \setminus N$).

Let $f \in C(X, E)$. Put

$$\varrho_*(f, F) = \operatorname*{ess\,sup}_{x \in X, y \in F(x)} \|f(x) - y\|. \tag{4.1}$$

By the latter we mean, as usual,

$$\inf_{N \in \mathcal{N}} \sup_{x \in X \setminus N, y \in F(x)} \|f(x) - y\|. \tag{4.2}$$

Now we define a distance of F from $C(X, E)$ by

$$\operatorname{dist}\big(F, C(X, E)\big) = \inf_{f \in C(X, E)} \varrho_*(f, F). \tag{4.3}$$

In this section we are interested in the following question: if $\operatorname{dist}\big(F, C(X, E)\big)$ is finite, does there exist an f_0 for which the infimum in (4.3) is attained?

A particular case of this question has been recently answered in the affirmative by Holmes and Kripke [2], namely the case when F is a bounded function of X into E and E is one-dimensional. The theorem which follows gives an answer to the question in a more general case.

THEOREM 3. *Let X, E and μ be as described above. Suppose that F is a map of X into 2^E and assume it is locally μ-essentially bounded.*

Then there exists an $f_0 \in C(X, E)$ such that

$$\varrho_*(f_0, F) = \operatorname{dist}\big(F, C(X, E)\big). \tag{4.4}$$

Theorem 3 is a consequence of two lemmas given below and of Theorem 1.

LEMMA 1. *Define*

$$F_*(x) = \bigcap_{U \in \mathscr{B}(x)} \bigcap_{N \in \mathscr{N}} \overline{\bigcup_{y \in U \setminus N} F(y)}, \tag{4.5}$$

where $\mathscr{B}(x)$ stands for a neighborhood base at x, $\mathscr{N}$ is the family of μ-null subsets of X and the bar indicates the closure.

Then F_* is a u.s.c. map of X into $K(E)$.

Proof. Consider the family (for an $x \in X$ fixed)

$$\{F_{U,N}\} = \Big\{\overline{\bigcup_{y \in U \setminus N} F(y)}\Big\} \quad \text{if} \quad U \in \mathscr{B}(x) \text{ and } N \in \mathscr{N}. \tag{4.6}$$

Family (4.6) has the finite intersection property, that is, any finite subfamily has a non-empty intersection. Since F is assumed to be locally essentially bounded and since E is finite-dimensional, there is a member of (4.6) which is compact, and without any loss of generality we may assume that all members of (4.6) are compact and contained in a fixed compact ball. Then by the finite intersection property, family (4.6) has a non-empty intersection which is exactly the set $F_*(x)$ given by (4.5). Hence (4.5) defines a map of X into $K(E)$. Let us now take an open subset G of E and suppose $F_*(x) \subset G$ for an $x \in X$. Again by the finite intersection property there is an $F_{U,N} \supset F_*(x)$ and $F_{U,N} \subset G$. The latter together with (4.5) implies that $F_*(x) \subset G$ for each $x \in U$. Thus we have proved that if an x_0 belongs to the set $A = \{x \mid F(x) \subset G\}$, where $G \subset E$ is open, then there is an open $U \subset X$ such that $x_0 \in U \subset A$, whence A is open and F_* is u.s.c., which was to be proved.

LEMMA 2. *If F is locally essentially bounded and F_* is defined by* (4.5), *then we have the inequality*

$$\varrho(f, F_*) \geqslant \operatorname{dist}\big(F, C(X, E)\big) \geqslant r(F_*) \quad \textit{for each } f \in C(X, E), \tag{4.7}$$

where r is defined by (1.2).

Proof. Let us fix $f \in C(x, E)$, $x_0 \in X$ and $\varepsilon > 0$. Put $\varrho(f, F_*) = \varrho_0$. By (0.1) and (4.5) there is an $U_0 \in \mathscr{B}(x_0)$ and $N_0 \in \mathscr{N}$ such that

$$F_*(x) \subset F_{U_0, N_0} \subset \bar{B}\big(f(x), \varrho_0 + \varepsilon\big) \quad \text{if } x \in U_0. \tag{4.8}$$

It follows from formula (4.8) that $\sup \|f(x) - y\| \leqslant \varrho_0 + \varepsilon$, where the supremum is taken for $x \in U_0 \setminus N_0$ and $y \in F(x)$, which in turn implies that $\varrho_*(f, F) \leqslant \varrho_0 + \varepsilon$ (cf. (4.1) and (4.2)). Since ε is arbitrary, we have $\varrho(f, F_*) \geqslant \varrho_*(f, F)$ and the first part of inequality (4.7) follows.

On the other hand, by (4.8) and (1.1) it is easy to see that

$$\sup_{x \in U_0 \setminus N_0, y \in F(x)} \|f(x) - y\| \geqslant r(x, F_*) \quad \text{if } x \in U_0.$$

Therefore by (4.1) and (4.2) we get

$$\varrho_*(f, F) \geqslant \sup_{x \in X} r(x, F_*) = r(F_*).$$

Hence, by (4.3), $\operatorname{dist}\big(F, C(F, X)\big) \geqslant r(F_*)$ and the proof of Lemma 2 is completed.

Proof of Theorem 3. It follows from Lemma 1 and Theorem 1 that there exists an $f_0 \in C(X, E)$ such that $\varrho(f_0, F_*) = r(F_*)$. Using inequality (4.7) of Lemma 2 we see that the same f_0 satisfies (4.4), which was to be proved.

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