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WEAK AUTOMORPHISMS OF LINEAR SPACES AND OF SOME OTHER ABSTRACT ALGEBRAS

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The purpose of this paper is a description of weak automorphisms of linear and affine spaces of v-algebras of dimension $\geqslant 3$ (see [1] and [3]). Some of our results were obtained independently by J. R. Senft in a forthcoming paper (e.g. Proposition (0) and a theorem similar to Theorem 1.2). Let $\mathfrak{A} = \langle A; \boldsymbol{F}\rangle$ be an algebra (cf. [2]). A permutation τ of the set A is called a *weak automorphism* of the algebra $\mathfrak{A}$ if the mapping $\tau^*: f \to f^*$ defined by the formula

$$f^*(x_1, \ldots, x_n) = \tau f(\tau^{-1}x_1, \ldots, \tau^{-1}x_n)$$

(cf. [1]) is a permutation of the set $\boldsymbol{A}(\boldsymbol{F})$ of all algebraic operations of the algebra $\mathfrak{A}$. We denote in the sequel the set of all automorphisms (respectively, weak automorphisms) by $\mathrm{Aut}(A)$ or $\mathrm{Aut}(\mathfrak{A})$ (respectively, $\mathrm{Aut}^*(\mathfrak{A})$). If two algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, we write $\mathfrak{A} \cong \mathfrak{B}$. The group of all permutations of a set X will be denoted by $S(X)$.

Let us observe that if τ is a permutation of A, then the identity

$$f^*(x_1, \ldots, x_n) = \tau f(\tau^{-1}x_1, \ldots, \tau^{-1}x_n)$$

is equivalent to

$$\tau^{-1}f^*(\tau x_1, \ldots, \tau x_n) = f(x_1, \ldots, x_n).$$

Therefore, if $\alpha \in \mathrm{Aut}(\mathfrak{A})$ and $\tau \in \mathrm{Aut}^*(\mathfrak{A})$, then

$$\begin{aligned}\tau^{-1}\alpha\tau f(x_1, \ldots, x_n) &= \tau^{-1}\alpha f^*(\tau x_1, \ldots, \tau x_n) \\ &= \tau^{-1}f^*(\alpha\tau x_1, \ldots, \alpha\tau x_n) = f(\tau^{-1}\alpha\tau x_1, \ldots, \tau^{-1}\alpha\tau x_n).\end{aligned}$$

Hence we have the following proposition:

(0) *For any algebra* $\mathfrak{A}$, *the group* $\mathrm{Aut}(\mathfrak{A})$ *is a normal subgroup of* $\mathrm{Aut}^*(\mathfrak{A})$.

1. Linear spaces. Let V be a linear space over the field Λ. One can consider it as an algebra

$$\mathfrak{V} = \langle V; +, \{\lambda x\}_{\lambda\in\Lambda}\rangle.$$

We assume in the sequel that $V \neq \{0\}$.

THEOREM 1.1. *Let $B = \{a_i\}_{i\in I}$ be a fixed basis of a linear space $\mathfrak{V}$. Then the mapping*

$$\mathrm{Aut}^*(\mathfrak{V}) \ni \tau \to \langle a_\tau, \varphi_\tau\rangle \in \mathrm{Aut}(\mathfrak{V}) \times \mathrm{Aut}(\Lambda),$$

where

$$a_\tau(a_i) = \tau(a_i) \quad \text{for } a_i \in B$$

and

$$(\varphi_\tau\lambda)x = \tau\big(\lambda(\tau^{-1}x)\big) \quad \text{for } \lambda\in\Lambda,\ x\in V,$$

establishes a one-to-one correspondence between $\mathrm{Aut}^*(\mathfrak{V})$ *and* $\mathrm{Aut}(\mathfrak{V})\times\mathrm{Aut}(\Lambda)$. *Therefore τ is a weak automorphism of $\mathfrak{V}$ if and only if it is of the form*

$$(*) \qquad \tau(x) = \sum_{k=1}^{n} (\varphi\lambda_k)\, a(a_{i_k}),$$

where $x = \sum_{k=1}^{n} \lambda_k a_{i_k}$, $a\in\mathrm{Aut}(\mathfrak{V})$, $\varphi\in\mathrm{Aut}(\Lambda)$.

Proof. First we prove that if $\tau\in\mathrm{Aut}^*(\mathfrak{V})$, then τ has the form $(*)$. Observe that every n-ary algebraic operation of the algebra $\mathfrak{V}$ has the form $\sum_{k=1}^{n}\lambda_k x_k$, where $\lambda_k\in\Lambda$ and 0 is the only algebraic constant of this algebra. Thus $\tau(0) = 0$ and $\tau(x+y) = \lambda_1\tau(x)+\lambda_2\tau(y)$. Putting firstly $x = 0$ and secondly $y = 0$ in the last equality, we obtain

$$(1.1) \qquad \tau(x+y) = \tau(x)+\tau(y).$$

Further, for every $\lambda\in\Lambda$ ($\lambda\neq 0$) there exists exactly one element $\lambda^*\in\Lambda$ such that $\tau(\lambda x) = \lambda^*\tau(x)$. Let us put

$$(1.2) \qquad \varphi_\tau(\lambda) = \lambda^* \text{ for } \lambda\neq 0, \text{ and } \varphi_\tau(0) = 0.$$

Obviously, $\varphi_\tau\in S(\Lambda)$, since the mapping $f\to f^*$ is a permuatation of the set $A^{(n)}(\mathfrak{A})$ for $n = 0, 1, 2, \ldots$ We show that $\varphi_\tau\in\mathrm{Aut}(\Lambda)$.

Indeed,

$$\tau\big((\lambda_1\lambda_2)x\big) = \varphi_\tau(\lambda_1\lambda_2)\tau(x) = \tau\big(\lambda_1(\lambda_2 x)\big) = \varphi_\tau(\lambda_1)\tau(\lambda_2 x) = \varphi_\tau(\lambda_1)\varphi_\tau(\lambda_2)\tau(x),$$

whence

$$\varphi_\tau(\lambda_1\lambda_2) = \varphi_\tau(\lambda_1)\varphi_\tau(\lambda_2).$$

Using (1.1) we analogously prove that

$$\varphi_\tau(\lambda_1+\lambda_2) = \varphi_\tau(\lambda_1)+\varphi_\tau(\lambda_2).$$

Let an element $x \in V$ be of the form

$$x = \sum_{k=1}^{n} \lambda_k a_{i_k}, \quad \text{where } a_{i_k} \in B.$$

From (1.1) and (1.2) we conclude that

$$\tau(x) = \sum_{k=1}^{n} \varphi_\tau(\lambda_k)\tau(a_{i_k}),$$

where $\{\tau(a_i)\}_{i\in I}$, is a basis for the space V. But the mapping $a_i \to \tau(a_i)$ can be uniquely extended to an automorphism a_τ of the algebra $\mathfrak{V}$ such that $a_\tau(a_i) = \tau(a_i)$ for $i \in I$. Hence

$$\tau(x) = \sum_{k=1}^{n} \varphi_\tau(\lambda_k) a_\tau(a_{i_k}),$$

and τ is of the form (*).

In order to show that the mapping τ defined by (*) is a weak automorphism of the algebra $\mathfrak{V}$ it is enough to verify the conditions $\tau \in \mathrm{Aut}(\langle V; +, -\rangle)$ and $\tau(\lambda x) = \varphi(\lambda)\tau(x)$, where $\varphi \in S(\Lambda)$. We verify, for example, the second condition. Let $x = \sum_{k=1}^{n} \lambda_k a_{i_k}$. Then

$$\tau(\lambda x) = \tau\Big(\sum_{k=1}^{n} (\lambda\lambda_k) a_{i_k}\Big) = \sum_{k=1}^{n} \varphi(\lambda\lambda_k) a(a_{i_k}) = \sum_{k=1}^{n} \varphi(\lambda)\varphi(\lambda_k) a(a_{i_k}) = \varphi(\lambda)\tau(x).$$

In a similar manner one can prove the other part of the theorem.

THEOREM 1.2. *The group* $\mathrm{Aut}^*(\mathfrak{V})$ *is the normal product of the group* $\mathrm{Aut}(\mathfrak{V})$ *and of the group* $\mathrm{Aut}(\Lambda)$.

Proof. From Theorem 1.1 we know that the correspondence $\tau \leftrightarrow (a_\tau, \varphi_\tau)$ is one-to-one. Let a be an automorphism of V such that

$$a(a_i) = \sum_{k=1}^{n} \lambda_k a_{i_k}, \quad \text{where } a_i, a_{i_k} \in B.$$

Let a^φ be the linear extension of the mapping

$$a_i \to \sum_{k=1}^{n} \varphi(\lambda_k) a_{i_k}, \quad \text{where } i \in I \text{ and } \varphi \in \mathrm{Aut}(\Lambda).$$

It is not hard to check that a^φ is an automorphism of V. Moreover, one can verify that the mapping $\varphi \to a^\varphi$ is a monomorphism of the group $\mathrm{Aut}(\Lambda)$ into the group $\mathrm{Aut}(\mathrm{Aut}\, V)$. It is also clear that if $H(\tau_1) = \langle a_1, \varphi_1\rangle$ and $H(\tau_2) = \langle a_2, \varphi_2\rangle$, then $H(\tau_1\tau_2) = \langle \bar{a}, \varphi_1\varphi_2\rangle$, where $\bar{a} \in \mathrm{Aut}(V)$.

We show that $\bar{a} = a_1 a_2^{\varphi_1}$. Indeed, let

$$a_2(a_i) = \sum_{k=1}^{n} \lambda_k a_{i_k}.$$

Then

$$a_2^{\varphi_1}(a_i) = \sum_{k=1}^{n} \varphi_1(\lambda_k) a_{i_k} \quad \text{and} \quad a_1 a_2^{\varphi_1}(a_i) = \sum_{k=1}^{n} (\varphi_1 \lambda_k) a_1(a_{i_k}).$$

But

$$\bar{a}(a_i) = \tau\big(a_2(a_i)\big) = \tau\Big(\sum_{k=1}^{n} \lambda_k a_{i_k}\Big) = \sum_{k=1}^{n} \varphi_1(\lambda_k) a_1(a_{i_k}).$$

Since $\bar{a}$ and $a_1 a_2^{\varphi_1}$ are identical on the vectors of the basis B, they are identical everywhere, and the theorem follows.

From this theorem follow immediately:

COROLLARY 1. *We have*

$$\mathrm{Aut}^*(V)/\mathrm{Aut}(V) = \mathrm{Aut}(\Lambda).$$

COROLLARY 2. *If V is a vector space over a prime field Λ, then*

$$\mathrm{Aut}^*(V) = \mathrm{Aut}(V).$$

The following corollary follows from the fact that the mapping $\varphi \to a^{\varphi}$ is a monomorphism:

COROLLARY 3. *The group* $\mathrm{Aut}^*(V)$ *is the direct product of the groups* $\mathrm{Aut}(V)$ *and* $\mathrm{Aut}(\Lambda)$ *if and only if the field Λ has only trivial automorphism.*

2. Affine spaces. Let V be an affine space over the field Λ. One can consider V as an algebra

$$\mathfrak{V}^0 = \langle V; \{\lambda x + \mu y\}, \text{ where } \lambda, \mu \in \Lambda \text{ and } \lambda + \mu = 1\rangle.$$

Of course, all algebraic n-ary operations of $\mathfrak{V}^0$ are of the form $\sum_{k=0}^{n} \lambda_k x_k$, where $\sum_{k=1}^{n} \lambda_k = 1$, $\lambda_k \in \Lambda$, and $n = 1, 2, 3 \ldots$ Theorems for a linear space are true (after a modification) also for an affine space.

As it is known, an arbitrary automorphism of an affine space is of the form $\varphi + c$, where φ is an automorphism of the linear space V and c is a fixed element of V. There was a conjecture of S. Fajtlowicz that weak automorphisms of the algebra $\mathfrak{V}^0$ have analogous form.

THEOREM 2.1. *A mapping ϱ of V onto itself is a weak automorphism of the algebra $\mathfrak{V}^0$ if and only if it has the form $\varrho = \tau + c$, where $\tau \in \mathrm{Aut}^*(\mathfrak{V})$ and $c \in V$.*

Proof. Let $\varrho \in \mathrm{Aut}^*(\mathfrak{V}^0)$. Then

$$\varrho(x+y-z) = \lambda\big(\varrho(x) + \varrho(y)\big) + \mu\varrho(z), \tag{2.1}$$

where $2\lambda + \mu = 1$. For $z = y$ we have $\varrho(x) = \lambda\varrho(x) + (\lambda + \mu)\varrho(y)$, whence $\lambda = 1$ and $\mu = -1$. Substituting these values together with $z = 0$ into (2.1) we obtain

$$\varrho(x+y) = \varrho(x) + \varrho(y) - \varrho(0). \tag{2.2}$$

For an arbitrary $\lambda \in \Lambda$ there exists exactly one element $\lambda^* \in \Lambda$ such that

(2.3) $$\varrho\big(\lambda x+(1-\lambda)y\big) = \lambda^* \varrho(x)+(1-\lambda^*)\varrho(y).$$

Setting

$$\varphi_\varrho(\lambda) = \lambda^* \quad \text{for } \lambda \in \Lambda$$

it is easy to see that $\varphi_\varrho \in S(\Lambda)$. Putting $y = 0$ in (2.3) we get (after a calculation)

(2.4) $$\varrho(\lambda x) = \varphi_\varrho(\lambda)\tau_\varrho(x)+\varrho(0),$$

where $\tau_\varrho(x) = \varrho(x)-\varrho(0)$.

Now we prove that $\tau_\varrho \in \mathrm{Aut}^*(\mathfrak{V})$. Indeed, $\tau_\varrho \in S(V)$, since $\varrho \in S(V)$ and, according to equality (2.2), we have

$$\tau_\varrho(x+y) = \varrho(x+y)-\varrho(0) = \big(\varrho(x)-\varrho(0)\big)+\big(\varrho(y)-\varrho(0)\big) = \tau_\varrho(x)+\tau_\varrho(y).$$

Moreover, we have by (2.4)

$$\tau_\varrho(\lambda x) = \varrho(\lambda x)-\varrho(0) = \varphi_\varrho(\lambda)\tau_\varrho(x).$$

In order to prove the converse implication of the theorem suppose that $\tau \in \mathrm{Aut}^*(\mathfrak{V})$ and $c \in V$. Then $\varrho = \tau+c \in S(V)$ since $\tau \in S(V)$. Hence we get $\varrho^{-1}(x) = \tau^{-1}(x)-\tau^{-1}(c)$. Further, if $f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k$, $\sum_{k=1}^{n} \lambda_k = 1$, then we have

$$\varrho\big(f(\varrho^{-1}(x_1), \ldots, \varrho^{-1}(x_n))\big) = \varrho\Big(\sum_{k=1}^{n}\lambda_k \varrho^{-1}(x_k)\Big) = \tau\Big(\sum_{k=1}^{n}\lambda_k\big(\tau^{-1}(x_k)-\tau^{-1}(c)\big)\Big)+c$$

$$= \sum_{k=1}^{n}\varphi_\tau(\lambda_k)\tau\big(\tau^{-1}(x_k)-\tau^{-1}(c)\big)+c = \sum_{k=1}^{n}\varphi_\tau(\lambda_k)(x_k-c)+c = \sum_{k=1}^{n}\varphi_\tau(\lambda_k)x_k.$$

Since $\sum_{k=1}^{n} \lambda_k = 1$, we deduce that $\varrho^{-1} \circ f \circ \varrho$ is an algebraic operation of the algebra $\mathfrak{V}^0$. One can easily show that the mapping $f \to f^* = \varrho \circ f \circ \varrho^{-1}$ is a permutation of the set of all algebraic operations of the algebra $\mathfrak{V}^0$. The theorem is thus proved.

Similarly to theorem (1.2) one can show the following

THEOREM 2.2. *The group* $\mathrm{Aut}^*(\mathfrak{V}^0)$ *is the normal product of the groups* $\mathrm{Aut}(\mathfrak{V})$ *and* $\mathrm{Aut}(\Lambda)$.

3. Generalized linear spaces. Let V be a linear space over the field Λ and let W be a linear subspace of V. Then the algebra

$$\mathfrak{V}_W = \langle V;+, \{\lambda x\}_{\lambda \in \Lambda}, \{a\}_{a \in W}\rangle$$

is called a *generalized linear space*. Obviously, if $W = \{0\}$, then $\mathfrak{V}_W$ is an ordinary linear space. Observe that every n-ary algebraic operation f of the algebra $\mathfrak{W}_W$ is of the form $\sum_{k=1}^{n} \lambda_k x_k + a$, where $a \in W$ and $A^{(0)}(\mathfrak{W}_W) = W$.

Let $\{b_i\}_{i\in I}$ and $\{b_i^0\}_{i\in I^0}$ be bases for spaces V and W, respectively. Let also $\varphi \in \operatorname{Aut} \Lambda$, $a \in \operatorname{Aut}(\langle V; +, \{\lambda x\}_{\lambda\in\Lambda}\rangle)$, $\beta \in \operatorname{Aut}(\langle W; +, \{\lambda x\}_{\lambda\in\Lambda}\rangle)$, and

$$x = \sum_{k=1}^{n} \chi_k b_{i_k} + \sum_{j=1}^{m} \mu_j b_{i_j}^0.$$

In a way similar to the proof of Theorem 1.1 one can prove

THEOREM 3.1. *The mapping τ is a weak automorphism of the algebra $\mathfrak{W}_W$ if and only if*

$$\tau(x) = \sum_{k=1}^{n} \varphi(\lambda_k) a(b_{i_k}) + \sum_{j=1}^{m} \varphi(\mu_j) \beta(b_{i_j}^0). \tag{3.1}$$

THEOREM 3.2. *There are monomorphisms h_1:* $\operatorname{Aut}^*(\mathfrak{V}_W) \to \operatorname{Aut}(\Lambda)$ *and h_2:* $\operatorname{Aut}^*(\mathfrak{V}_W) \to \operatorname{Aut}^*(\langle W; +, \{\lambda x\}_{\lambda\in\Lambda}\rangle)$ *such that*

$$\ker h_1 \cap \ker h_2 = \operatorname{Aut}(\mathfrak{V}_W). \tag{**}$$

Proof. In order to find h_1 it is enough to observe that, similarly to the proof of Theorem 1.1, the mapping defined by the formula $\varphi_\tau(0) = 0$ and $\varphi_\tau(\lambda) = \lambda^*$ is an automorphism of the field Λ, and that $\operatorname{Aut}(\Lambda)$ is the homomorphic image of $\operatorname{Aut}^*(\mathfrak{V}_W)$ under the homomorphism $h_1(\tau) = \varphi_\tau$.

In order to find h_2 it is enough to notice that the mapping $\tau_0 = \tau | W$ is a weak automorphism of the algebra $\langle W; +, \{\lambda x\}_{\lambda\in\Lambda}\rangle$ and that the mapping $h_2(\tau) = \tau_0$ is a homomorphism of the group $\operatorname{Aut}^*(\mathfrak{V}_W)$ onto the group $\operatorname{Aut}(\langle W; +, \{\lambda x\}_{\lambda\in\Lambda}\rangle)$. Equality $(**)$ is obvious.

4. Generalized affine spaces. Let V be a linear space over the field Λ and let W be a subspace of V. The algebra

$$\mathfrak{V}_W^0 = \langle V; \{\lambda x + \mu y + a\}, \text{ where } \lambda, \mu \in \Lambda, \lambda + \mu = 1 \text{ and } a \in W\rangle$$

is called a *generalized affine space*.

We give an idea of a proof of the following

THEOREM 4.1. *A mapping σ is a weak automorphism of the algebra $\mathfrak{V}_W^0$ if and only if*

$$\sigma = \varrho + c, \tag{4.1}$$

where $\varrho \in \operatorname{Aut}^*(\mathfrak{V}_W)$ and $c \in V$.

Proof. If $\sigma \in \operatorname{Aut}^*(\mathfrak{V}_W^0)$, then $\sigma(x+y) = \sigma(x) + \sigma(y) - \sigma(0)$ and $\sigma\big(\lambda x + (1-\lambda) y\big) = \lambda^* \sigma(x) + (1-\lambda^*)\sigma(y) + a^*$. Hence $\sigma(\lambda x) = \varphi_\sigma(\lambda)\big(\sigma(x) - \sigma(0)\big)$, where $\varphi_\sigma(\lambda) = \lambda^*$. Let $\sigma_\sigma(x) = \sigma(x) - \sigma(0)$. Then $\varrho_\sigma \in S(V)$ and $\varrho_\sigma | W \in S(W)$. It is sufficient to show that

$$\varrho_\sigma \in \mathrm{Aut}^*(\langle V; +, \{\lambda x\}_{\lambda \in \Lambda}\rangle).$$

With this in view verify that there is $\varrho_\sigma(x+y) = \sigma(x+y) - \sigma(0) = (\sigma(x) - \sigma(0)) + (\sigma(y) - \sigma(0)) = \varrho_\sigma(x) + \varrho_\sigma(y)$.

It is easy to werify that $\varrho_\sigma(\lambda x) = \varphi_\sigma(\lambda)\varrho_\sigma(x)$. Obviously, if $\varrho \in \mathrm{Aut}^*(\mathfrak{V}_W)$ and $c \in V$, then $\sigma = \varrho + c$ is a weak automorphism of the algebra $\mathfrak{V}_W^0$ (cf. the proof of Theorem 2.1).

5. Unary abstract algebras having bases. Let $\mathfrak{A} = \langle A; \boldsymbol{F}\rangle$ be a unary algebra. The set $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(1)}(\mathfrak{A})$ of all algebraic operations can be considered as semigroup with identity and with a binary operation $\cdot$ (the superposition of algebraic operations). We assume that $\mathfrak{A}$ has a basis (cf. [2]).

THEOREM 5.1. *If $\mathfrak{A}$ is a unary algebra having a basis, then $\tau \in \mathrm{Aut}^*(\mathfrak{A})$ if and only if τ is of the form*

$$\tau(x) = (\varphi f)(a(b)), \tag{5.1}$$

where $x = f(b)$, b is an element of a basis B, $a \in \mathrm{Aut}\,\mathfrak{A}$, and $\varphi \in \mathrm{Aut}(\langle \boldsymbol{A}; \cdot\rangle)$.

Proof. Let B be a basis of the algebra $\mathfrak{A}$. Then for $x \in A$ we have $x = f(b)$ for some $b \in B$. The mapping induced by $b \to \tau(b)$ can be uniquely extended to an automorphism a_τ of the algebra $\mathfrak{A}$. So we get $\tau(x) = \tau(f(b)) = f^*(\tau(b)) = f^*(a_\tau(b))$ and the mapping $f \to f^*$ is an automorphism of the semigroup $\langle \boldsymbol{A}^{(1)}; \cdot\rangle$. The opposite way is quite obvious.

THEOREM 5.2. *If $\mathfrak{A}$ is a unary algebra having a basis, then the group* $\mathrm{Aut}^*(\mathfrak{A})$ *is the normal product of the groups* $\mathrm{Aut}(\mathfrak{A})$ *and* $\mathrm{Aut}(\langle \boldsymbol{A}^{(1)}; \cdot\rangle)$.

Proof. Let B be a fixed basis of $\mathfrak{A}$. We have already shown that there is a 1-1 correspondence between all weak automorphisms of $\mathfrak{A}$ and all pairs $\langle a, \varphi\rangle$, where $a \in \mathrm{Aut}(\mathfrak{A})$ and $\varphi \in \mathrm{Aut}(\langle \boldsymbol{A}^{(1)}; \cdot\rangle)$. If $a \in \mathrm{Aut}(\mathfrak{A})$ and $\varphi \in \mathrm{Aut}(\langle \boldsymbol{A}^{(1)}; \cdot\rangle)$, then we denote by a^φ the mapping $A \to A$ defined by

$$a^\varphi(b) = \varphi(f)(b') \quad \text{if } a(b) = f(b') \text{ and } b, b' \in B,$$
$$a^\varphi(g(b)) = g(a^\varphi(b)) \quad \text{for all } g \in \boldsymbol{A}.$$

One can check that a^φ is an automorphism of $\mathfrak{A}$ and that $a_1 \neq a_2$ implies $a_1^\varphi \neq a_2^\varphi$. If $\xi\colon b \to f(b')$ is an automorphism of $\mathfrak{A}$, then $\xi = a^\varphi$ provided a is an automorphism with $a(b) = (\varphi^{-1}f)(b')$, the mapping a^φ is a permutation of the set Aut $\mathfrak{A}$. We also have $1^\varphi = 1$ and $(\alpha\beta)^\varphi = \alpha^\varphi\beta^\varphi$. Consequently, the mapping $\varphi \to a^\varphi$ is a homomorphism of $\mathrm{Aut}(\langle \boldsymbol{A}^{(1)}; \cdot\rangle)$ into the group $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{A}))$ (it is even a monomorphism). Clearly, if $\tau_1 = \langle a_1, \varphi_1\rangle$ and $\tau_2 = \langle a_2, \varphi_2\rangle$, then $\tau_1\tau_2 = \langle a, \varphi_1\varphi_2\rangle$. But if $a_2(b) = f(b')$, then $a(b) = \tau_1\tau_2(b) = \tau_1(f(b')) = (\varphi_1(f))(a_1(b'))$ and $a_1 a_2^{\varphi_1}(b) = a_1(\varphi_1 f)(a_1 b') = (\varphi_1 f)(a_1(b'))$. Therefore $a = a_1 a_2^{\varphi_1}$ and the theorem follows.

6. v-**algebras.** A theorem on a representation of v-algebras due to K. Urbanik (cf. [3]) implies that every v-algebra of dimension $\geqslant 3$ is one of the algebras considered in sections 3, 4 and 5 of the present paper. Therefore we infer from Theorems 3.1, 4.1, and 5.1 the following

THEOREM. *Every weak automorphism of a v-algebra of dimension* $\geqslant 3$ *has one of the forms* (3.1), (4.1) *or* (5.1).

Thus weak automorphisms of a v-algebra of dimension $\geqslant 3$ are completely described.

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