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# *CONVOLUTION OF $L^2$-FUNCTIONS*

BY

NEIL W. RICKERT (PRINCETON, N. J.)

In [5], [6], [7], and [10] the question has been studied of whether, for $p > 1$, $L^p(G)$ is a convolution algebra ($G$ being a locally compact group). The obvious conjecture has been that this is only true if $G$ is compact. However the problem has not as yet yielded to a complete solution. For $p > 2$ the conjecture has been proved in [5], [8], and [10]. For $p = 2$ it has also been proved for totally disconnected groups. For all $p > 1$ the conjecture has been proved for solvable groups. In this note we propose to settle the question for $p = 2$.

LEMMA 1. *Let $G$ be a connected matrix Lie group whose radical is not compact. Then there is a continuous positive definite function on $G$ which is not in $L^p$ for any $p < \infty$, but which is the uniform limit on compact sets of continuous positive definite functions with compact support.*

Proof. Takenouchi [9] has shown that if $G$ is a $(C)$-group, the function which is identically 1 is the uniform limit on compact sets of continuous positive definite functions with compact support. His technique was to prove it first for Abelian groups, and then to show how, having proved it for two groups, to prove it for the semi-direct product. But under the hypotheses of our lemma, $G$ is locally a semi-direct product of its radical, which is solvable (hence a $(C)$-group) and a semi-simple Lie group. Using the techniques of Takenouchi it is readily proved that there is a positive definite function on $G$ which is constant on cosets of the radical, and which is in the uniform closure on compact sets of continuous positive definite functions with compact support. The result follows.

LEMMA 2. *Let $G$ be a connected Lie group such that $L^p(G)$ is closed under convolution for some $p > 1$. Then the radical of $G$ is compact.*

Proof. As proved in [6] the centre of $G$ is compact. We may factor by the centre, without changing the hypotheses (see [5]), so we may assume

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that $G$ is a matrix group. We may also assume that $G$ is unimodular (see [7]). By suitably choosing the Haar measure on $G$ we may assume that $L^p(G)$ is a Banach algebra (see [5]). If $f$ is in $L^p$, then we define its adjoint to be the function $g$ defined by $\overline{g(x)} = f(x^{-1})$. It is easily seen that with this definition, $L^p(G)$ is an algebra with continuous involution. Let $U_n$ be decreasing compact symmetric neighborhoods of the identity in $G$ which form a neighborhood base. Denote by $f_n$ the characteristic function of $U_n$ divided by the measure of $U_n$. Thus $f_n * g \to g$ as $n \to \infty$ for $g$ in $L^p$. Assume that $\varphi$ is a continuous positive functional on $L^p$. By a slight modification of the argument on page 188 of [4] it is easily seen that

$$\|\varphi\| \leqslant \limsup |\varphi(f_n)|.$$

But this says exactly that if $h$ is a continuous positive definite function in $L^q(G)$ (where $1/p+1/q = 1$), then $\|h\| \leqslant h(1)$. Applying Lemma 1, we conclude that the radical of $G$ is compact.

To prove our main result, it is now seen that we need only prove it for semi-simple Lie groups. We may further assume that such groups have a compact, hence finite, centre (see [6]). This leads us to a consideration of the associated symmetric space. Thus assume that $G$ is a non-compact semi-simple Lie group with finite centre, and denote by $K$ a maximal compact subgroup. The space $G/K$ is a Riemannian symmetric space (see [1]). If $f$ is a function on $G$ such that $f(kxk') = f(x)$ for $k, k'$ in $K$, $f$ corresponds to a function on the symmetric space which is $K$-invariant. Many of the usual theorems of functional analysis apply to such functions (see [1]). In particular, such functions in $L^1 \cap L^2$ form a commutative algebra under convolution, and the Plancherel theorem is valid for some suitable Plancherel measure on the maximal ideal space of the algebra. Harish-Chandra has determined the Plancherel measure in [2] and [3]. In particular, it follows from his work that there are sets of arbitrarily small positive measure. Thus in the maximal ideal space with the Plancherel measure, $L^2$ is not a Banach algebra under pointwise multiplication. It follows, because of the relationship of convolution on $G$ to pointwise multiplication in the Plancherel space, that the space of $K$-invariant $L^2$ functions on $G/K$ is not a Banach algebra under convolution. Thus, in particular, we have shown

LEMMA 3. *Let $G$ be a non-compact semi-simple Lie group. Then $L^2(G)$ is not a convolution algebra.*

It now easily follows from results in [5] that

THEOREM. *Let $G$ be a non-compact locally compact group. Then $L^2(G)$ is not a convolution algebra.*

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INSTITUTE FOR ADVANCED STUDY
YALE UNIVERSITY

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