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ADDITIVE FUNCTIONALS ON ORLICZ SPACES

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1. Introduction. The space $\mathfrak{X}^*$ (the dual space of $\mathfrak{X}$) of linear continuous functionals on linear metric semi-ordered space $\mathfrak{X}$ constitutes a subspace of the family $\mathfrak{X}^{\oplus}$ (the *add-dual spaec* of $\mathfrak{X}$) of additive functionals on the same space. The formal definition of such functionals will be given later on. Even if $\mathfrak{X}$ is a Banach space, in general there are additive functionals which are not linear. The more so it is also true: for linear metric spaces that are neither Banach nor even locally convex, and in general $\mathfrak{X}^* \subsetneqq \mathfrak{X}^{\oplus}$. For many spaces of measurable functions over the finite measure space $(S, \mathscr{B}, \mu)$ (it is fixed in all this paper) with the F-norm preserving the natural semi-order, the structure of their duals is well known. We only mention the classical Riesz' results concerning both the spaces of continuous functions and the spaces $\mathfrak{L}_p(S, \mathscr{B}, \mu)$, $p \geqslant 1$. Also the Riesz-type theorem of Birnbaum and Orlicz [1] describing the dual to the Orlicz space $\mathfrak{L}_{\Phi}(\Phi)$ is a convex Orlicz function satisfying the Δ_2-condition), as the space $\mathfrak{L}_{\Psi}$, where Ψ is complementary to Φ in the sense of Young, is of our interest. On the other hand, there are a number of results explaining the structure of some add-dual spaces. We only mention the results of Martin and Mizel [8] concerning add-dual to the space of all bounded measurable functions over a finite atom-free measure space, and results of Friedman and Katz [4] concerning the Banach space of real-valued continuous functions on a compact metric space. Nevertheless, the case of non-locally convex spaces with no non-trivial linear continuous functionals, as far as I know, has not yet been thoroughly investigated. It seems to be especially interesting since then no additive functional is linear. Merely Friedman and Katz [5] have recently described $\mathfrak{L}_p^{\oplus}(S, \mathscr{B}, \mu)$ for $p > 0$. On the other hand, Day [3] has proved that for the spaces $\mathfrak{L}_p$, $0 < p < 1$, over an atomless measure space there exist no linear continuous functionals except the trivial ones. The same conclusion has been obtained for the space of all measurable functions over an atomless measure space (with the topology induced by convergence in measure) by Mazur and Orlicz [10], and for Orlicz spaces $\mathfrak{L}_\Phi$, where, by Gramsch [6],

$$\lim_{t\to\infty} [\Phi(t)/t] = 0.$$

The aim of this paper is to extend the Friedman-Katz description of $\mathfrak{L}_p^\oplus$ to the case of Orlicz spaces (Section 5). Simultaneously, our characterization of additive functionals seems also a lot simpler. Further (Section 6), by virtue of some theorems proved by Urbanik [13] and Urbanik and Woyczyński [14] connecting Orlicz spaces with the theory of random measures, certain probability applications of our representation theorem are indicated.

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2. Additive transformations and functionals. Let both $\mathfrak{X}$ and $\mathfrak{Y}$ be linear metric semi-ordered spaces with F-norms $\|\cdot\|_\mathfrak{X}$ and $\|\cdot\|_\mathfrak{Y}$, respectively. A mapping $\boldsymbol{A}$ of $\mathfrak{X}$ into $\mathfrak{Y}$ is said to be an *additive transformation* if $\boldsymbol{A}$ satisfies the following conditions:

(a) *Continuity.* For each $\varepsilon > 0$ and $b > 0$, there exists a $\delta = \delta(b, \varepsilon)$ such that the inequalities $\|\mathfrak{x}_1\|_\mathfrak{X} \leqslant b$, $\|\mathfrak{x}_2\|_\mathfrak{X} \leqslant b$ and $\|\mathfrak{x}_1 - \mathfrak{x}_2\|_\mathfrak{X} \leqslant \delta$ $(\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X})$ imply $\|\boldsymbol{A}\mathfrak{x}_1 - \boldsymbol{A}\mathfrak{x}_2\|_\mathfrak{Y} \leqslant \varepsilon$.

(b) *Boundedness.* For each $b > 0$, there exists a $B = B(b)$ such that $\|\mathfrak{x}\|_\mathfrak{X} \leqslant b$ $(\mathfrak{x} \in \mathfrak{X})$ implies $\|\boldsymbol{A}\mathfrak{x}\|_\mathfrak{Y} \leqslant B$, i.e.

$$\sup_{\|\mathfrak{x}\|_\mathfrak{X} \leqslant b} \|\boldsymbol{A}\mathfrak{x}\|_\mathfrak{Y} < \infty.$$

(c) *Additivity.* If $\mathfrak{x}_1$ and $\mathfrak{x}_2$ are disjoint elements of $\mathfrak{X}$ $(|\mathfrak{x}_1| \wedge |\mathfrak{x}_2| = 0)$, then $\boldsymbol{A}(\mathfrak{x}_1 + \mathfrak{x}_2) = \boldsymbol{A}\mathfrak{x}_1 + \boldsymbol{A}\mathfrak{x}_2$.

When $\mathfrak{Y} = R$ (space of reals) $\boldsymbol{A}$ is said to be an *additive functional.* The linear space of all additive functionals on $\mathfrak{X}$ will be denoted by $\mathfrak{X}^\oplus$, and its elements, as a rule, by $\mathfrak{x}^\oplus$, $\mathfrak{y}^\oplus$ and so on. The foregoing definitions are a modification of definitions given in [2], [4], [5] and [8].

3. Modular and Orlicz spaces. A linear semi-ordered space $\mathfrak{X}_M$ is said to be *modular* (in the sense of Musielak and Orlicz [11]) if a functional M, called the *modular*, satisfying the following four requirements, is defined on it:

I. $M(\mathfrak{x}) = 0$ if and only if $\mathfrak{x} = 0$;

II. $M(\mathfrak{x}) = M(-\mathfrak{x})$;

III. $M(\alpha\mathfrak{x} + \beta\mathfrak{y}) \leqslant M(\mathfrak{x}) + M(\mathfrak{y})$ for any $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$.

IV. $a_n \to 0$ implies $M(a_n\mathfrak{x}) \to 0$ for any $\mathfrak{x} \in \mathfrak{X}$.

Moreover, it becomes a complete linear metric space under the non-homogeneous F-norm

$$\|\mathfrak{x}\|_M = \inf\{c : c > 0,\ M(c^{-1}\mathfrak{x}) \leqslant c\}.$$

If $\mathfrak{X}$ is a modular space of measurable functions over a finite measure space $(S, \mathscr{B}, \mu)$ with the usual semi-order $\leqslant$ in it, then we shall assume that $M(\mathfrak{x}) \leqslant M(\mathfrak{y})$ whenever $\mathfrak{x} \leqslant \mathfrak{y}$, $\mathfrak{x}, \mathfrak{y} \geqslant 0$. M itself is then an additive functional on $\mathfrak{X}$ in the sense of Section 2.

As examples of modular spaces, important for our purposes, we indicate the so-called *Orlicz spaces*. In fact, consider a function $\Phi: R^+ \to R^+$ non-decreasing and continuous, vanishing only at zero, and tending to infinity as the argument approaches infinity. The class of all functions Φ satisfying the conditions mentioned above will be denoted by $\mathscr{K}$ and we will refer to its elements as to *Orlicz functions*. Only Orlicz functions Φ that satisfy Δ_2-condition will be of our interest. Recall that $\Phi \in \Delta_2$ if and only if for some constants $b > 0$ and $u_0 \geqslant 0$ we have $\Phi(2u) \leqslant b\Phi(u)$ for $u \geqslant u_0$. We still introduce the class $\mathscr{K}_0 \subset \mathscr{K}$ of all Orlicz functions Φ such that

$$[\Phi(u)/u^2] \leqslant c[\Phi(v)/v^2]$$

for $u \geqslant v \geqslant v_0$, where $c > 0$ and $v_0 \geqslant 0$ are some constants.

For every real $\mathscr{B}$-measurable function $\mathfrak{x}$ on the finite measure space $(S, \mathscr{B}, \mu)$ we put

$$\mathbf{I}_\Phi(\mathfrak{x}) = \int_S \Phi(|\mathfrak{x}(s)|)\mu(ds).$$

The set $\mathfrak{L}_\Phi(S, \mathscr{B}, \mu)$ of all $\mathscr{B}$-measurable functions $\mathfrak{x}$ for which $\mathbf{I}_\Phi$ is finite is a modular space with the modular $\mathbf{I}_\Phi$ and usual semi-order. This is what is called an Orlicz space.

The set $\mathscr{K}$ of all Orlicz functions can also be semi-ordered in the following manner. We say that $\Phi_1 \in \mathscr{K}$ is *non-weaker* than $\Phi_2 \in \mathscr{K}$, and write $\Phi_2 \precsim \Phi_1$, if for some constants $a, k > 0$ and $u_0 \geqslant 0$, we have $\Phi_2(u) \leqslant a\Phi_1(ku)$ for $u \geqslant u_0$. Φ_1 and Φ_2 are *equivalent* ($\Phi_1 \sim \Phi_2$) if $\Phi_1 \precsim \Phi_2$ and $\Phi_2 \precsim \Phi_1$. We know [9] that $\mathfrak{L}_\Phi = \mathfrak{L}_\Psi$ for $\Phi, \Psi \in K$, if and only if $\Phi \sim \Psi$. The equality $\mathfrak{L}_\Phi = \mathfrak{L}_\Psi$ is understood here as an identity of underlying sets together with equivalence of topologies induced by norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Psi$. It is also known that this topology is locally convex if and only if Φ is equivalent to some convex function from $\mathscr{K}$. The space $\mathfrak{L}_\Phi$ is then even a Banach space. Of course, we always identify functions which are equal μ-almost everywhere.

4. The simplest non-linear operator acting in Orlicz spaces. A real-valued function $K(t, s)$ defined on the Cartesian product $R \times S$ is said *to satisfy the Carathéodory conditions* if and only if it is continuous with respect to t for μ-almost all $s \in S$, and is $\mathscr{B}$-measurable with respect to s for every fixed t. By $\mathbf{K}$ (may be with indices) we denote the operator acting on real-valued functions $\mathfrak{x}(s)$, $s \in S$, by means of the formula

$$(\mathbf{K}\mathfrak{x})(s) = K(\mathfrak{x}(s), s).$$

It will be always supposed that $\mathbf{K}$ satisfies the Carathéodory conditions. Denote by $\mathfrak{B}_{\Phi}(r)$ the ball $\{\mathfrak{x}\colon \mathfrak{x}\in\mathfrak{L}_{\Phi},\ \|\mathfrak{x}\|_{\Phi}\leqslant r\}$.

THEOREM 4.1. *If* $\mathbf{K}0=0$, *then the following statements are equivalent*:

(i) $\mathbf{K}\colon \mathfrak{B}_{\Phi}(r)\to\mathfrak{L}_1(S,\mathscr{B},\mu)$ *for some* $r>0$;

(ii) $\mathbf{K}\colon \mathfrak{L}_{\Phi}(S,\mathscr{B},\mu)\to\mathfrak{L}_1(S,\mathscr{B},\mu)$;

(iii) $\mathbf{K}$ *is continuous at every point of* $\mathfrak{L}_{\Phi}(S,\mathscr{B},\mu)$;

(iv) $\mathbf{K}$ *is bounded, i.e., for every* $0\leqslant\varrho<\infty$,

$$\sup_{\mathfrak{x}\in\mathfrak{B}_{\Phi}(\varrho)}\|\mathbf{K}\mathfrak{x}\|_{\mathfrak{L}_1}<\infty;$$

(v) *the inequality* $|K(t,s)|\leqslant a\Phi(t)+\mathfrak{a}(s)$ *for* $-\infty<t<\infty$ *and* $s\in S$, *holds true with some* $a\geqslant 0$ *and* $\mathfrak{a}(s)\in\mathfrak{L}_1$.

Proof. The implications (v) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious, and it suffices to prove the following ones: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (v).

(i) $\Rightarrow$ (ii). Let $\mathfrak{x}\in\mathfrak{L}_{\Phi}$. By the absolute continuity of $\mathbf{I}_{\Phi}$ we can write $\mathfrak{x}(s)$ in the form

$$\mathfrak{x}(s)=\mathfrak{x}_0(s)+\mathfrak{x}_1(s)+\ldots+\mathfrak{x}_k(s),$$

where $\mathbf{I}_{\Phi}(\mathfrak{x}_i)<r'(r)$, $i=0,1,\ldots,k$, $r'(r)$ is sufficiently small for $\|\mathfrak{x}_i\|_{\Phi}<r$, and functions $\mathfrak{x}_i$ have disjoint supports for different indices $i=0,1,\ldots,k$. By our assumption

$$\mathbf{K}\mathfrak{x}_i(s)\mathbf{K}\mathfrak{x}_j(s)=0 \quad \text{for} \quad s\in S \text{ and } i\neq j,$$

and

$$\mathbf{K}\mathfrak{x}(s)=\mathbf{K}\mathfrak{x}_0(s)+\mathbf{K}\mathfrak{x}_1(s)+\ldots+\mathbf{K}\mathfrak{x}_k(s). \tag{1}$$

Since $\mathbf{K}\mathfrak{B}_{\Phi}(r)\subset\mathfrak{L}_1$, every function $\mathbf{K}\mathfrak{x}_i(s)$, $i=0,1,\ldots,k$, is an element of $\mathfrak{L}_1$, and by (1) and the linearity of $\mathfrak{L}_1$ so is $\mathbf{K}\mathfrak{x}(s)$.

(ii) $\Rightarrow$ (iii). In order to prove this we shall have to make use of the following

LEMMA 4.2 (cf. [7]). *Let* $\mathbf{K}_1\colon\mathfrak{L}_1\to\mathfrak{L}_1$. *Then, in the norm of* $\mathfrak{L}_1$, *the operator* $\mathbf{K}_1$ *is continuous, bounded and, moreover, for the function* $K_1(t,s)$ *the inequality*

$$|K_1(t,s)|\leqslant a_1|t|+\mathfrak{a}_1(s) \quad \textit{for} \quad -\infty<t<\infty \textit{ and } s\in S,$$

holds true with some non-negative constant a_1 *and* $\mathfrak{a}_1\in\mathfrak{L}_1$.

Now, we shall show that $\mathbf{K}$ is continuous at $0\in\mathfrak{L}_{\Phi}$. Suppose that $\mathfrak{x}_n\in\mathfrak{L}_{\Phi}$, $n=1,2,\ldots$, is a sequence of functions such that

$$\lim_{n\to\infty}\|\mathfrak{x}_n\|_{\Phi}=0 \quad (\Leftrightarrow \lim_{n\to\infty}\mathbf{I}_{\Phi}(\mathfrak{x}_n)=0). \tag{2}$$

This means that the sequence $\{\Phi(\mathfrak{x}_n)\}$ of $\mathfrak{L}_1$-functions converges to zero in $\mathfrak{L}_1$. The operator $\mathbf{K}_1$ defined by the formula $\mathbf{K}_1(\cdot) = \mathbf{K}\Phi^{-1}(\cdot)$ (here, and in the sequel, Φ^{-1} denotes the function inverse to Φ) obviously acts from $\mathfrak{L}_1$ into $\mathfrak{L}_1$, and utilizing Lemma 4.2 we infer that it is continuous at zero. Applying now this result to the sequence $\{\Phi(\mathfrak{x}_n)\}$ we see that

$$\lim_{n\to\infty} \int_S |\mathbf{K}_1 \Phi[\mathfrak{x}_n(s)]|\,\mu(ds) = \lim_{n\to\infty} \int_S |\mathbf{K}\mathfrak{x}_n(s)|\,\mu(ds) = 0.$$

This ends the proof, for the continuity of $\mathbf{K}$ at an arbitrary point $\mathfrak{x}_0 \in \mathfrak{L}_\Phi$ is equivalent to the continuity of the operator $\mathbf{K}_2\mathfrak{x} \equiv \mathbf{K}(\mathfrak{x}_0+\mathfrak{x})-\mathbf{K}\mathfrak{x}_0$ at $0 \in \mathfrak{L}_\Phi$.

(ii) $\Rightarrow$ (iv). It is easy to see that the operator $\mathbf{K}_3$ defined by the formula

$$\mathbf{K}_3\mathfrak{y}(s) \equiv \mathbf{K}\big(\varrho\Phi^{-1}|\mathfrak{y}(s)|\big), \qquad \varrho \geqslant 0,$$

acts from $\mathfrak{L}_1$ into $\mathfrak{L}_1$. By Lemma 4.2 we have

$$\sup_{\|\mathfrak{y}\|_1 \leqslant \varrho} \int_S |\mathbf{K}_3\mathfrak{y}(s)|\,\mu(ds) < \infty$$

for every $0 \leqslant \varrho < \infty$. Put $\mathfrak{y}(s) = \Phi\big(\mathfrak{x}(s)/\varrho\big)$. The inequality $\|\mathfrak{x}\|_\Phi \leqslant \varrho$ implies that $\mathbf{I}_\Phi\big(\mathfrak{x}(s)/\varrho\big) \leqslant \varrho$ (see e.g. [11], p. 52), and thus we get inequalities

$$\sup_{\|\mathfrak{x}\|_\Phi \leqslant \varrho} \int_S |\mathbf{K}\mathfrak{x}(s)|\,\mu(ds) \leqslant \sup_{\mathbf{I}_\Phi(\mathfrak{x}(s)/\varrho) \leqslant \varrho} \int |\mathbf{K}\mathfrak{x}(s)|\,\mu(ds) = \sup_{\|\mathfrak{y}\|_1 \leqslant \varrho} \int_S |\mathbf{K}_3\mathfrak{y}(s)|\,\mu(ds) < \infty,$$

which complete the proof.

(ii) $\Rightarrow$ (v). We remember that $\mathbf{K}_1(\mathfrak{y}) = \mathbf{K}\Phi^{-1}(\mathfrak{y})$ acts from $\mathfrak{L}_1$ into $\mathfrak{L}_1$ and, by virtue of Lemma 4.2, the inequality

$$\big|K\big(\Phi^{-1}(t), s\big)\big| \leqslant a|t| + \mathfrak{a}(s) \qquad \text{for} \qquad -\infty < t < \infty \text{ and } s \in S,$$

holds true with some $a \geqslant 0$ and $\mathfrak{a}(s) \in \mathfrak{L}_1$. Putting $\mathfrak{y}(s) = \Phi\big(\mathfrak{x}(s)\big)$ we get the thesis.

It should be still mentioned that in order to prove the equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) we do not need to assume that $\mathbf{K}0 = 0$.

5. Representation of additive functionals on Orlicz spaces.

THEOREM 5.1. $\mathfrak{x}^\oplus \in \mathfrak{L}_\Phi^\oplus$ *if and only if*

$$\mathfrak{x}^\oplus \mathfrak{x} = \int_S K\big(\mathfrak{x}(s), s\big)\,\mu(ds), \qquad \mathfrak{x} \in \mathfrak{L}_\Phi,$$

where

(a) $K(0, s) = 0$;

(b) *$K(t, s)$ satisfies the Carathéodory conditions;*

(c) *the inequality*

$$|K(t, s)| \leqslant a\Phi(t) + \mathfrak{a}(s) \quad \text{for} \quad -\infty < t < \infty \text{ and } s \in S,$$

holds true with some non-negative constant a and $\mathfrak{a} \in \mathfrak{L}_1$.

(d) **K** *is an additive transformation.*

Proof. We shall utilize the following lemma, similar to that of Friedman and Katz [5], Lemma 2. Its proof is essentially the same and we omit it.

LEMMA 5.2. *For every $\mathfrak{x}^{\oplus} \in \mathfrak{L}_{\Phi}^{\oplus}$ there exists a kernel $K'(t, s)$ and a function $\beta(s)$ such that*

(α) *$K'(0, s) = 0$;*

(β) *$K'(t, s)$ satisfies the Carathéodory conditions;*

(γ) *for each $b > 0$ there exists an $H = H(b)$ such that $|t| < b$ implies $|K'(t, s)| < N$ for μ-almost all $s \in S$;*

(δ) *$\beta(s) = d\mu_*/d\mu$ for some $\mu_* \sim \mu$;*

(ε) *$\mathfrak{x}^{\oplus}(h\chi_B) = \int_S K'\big(h\chi_B(s), s\big)\beta(s)\mu(ds)$ for $h \in R$, where χ_B is the characteristic function of the set $B \in \mathscr{B}$.*

Now we put $K(t, s) = K'(t, s)\beta(s)$. Obviously, $K(t, s)$ satisfies (a)-(c) of Theorem 5.1. For each $\mathfrak{x} \in \mathfrak{L}_{\Phi}$ define $\mathfrak{x}_1^{\oplus}\mathfrak{x}$ by

$$\mathfrak{x}_1^{\oplus}\mathfrak{x} = \int_S K\big(\mathfrak{x}(s), s\big)\mu(ds). \tag{3}$$

It remains to prove that $\mathfrak{x}^{\oplus}\mathfrak{x} = \mathfrak{x}_1^{\oplus}\mathfrak{x}$ for all $\mathfrak{x} \in \mathfrak{L}_{\Phi}$. In fact, these functionals, being additive, continuous and bounded on $\mathfrak{L}_{\Phi}$, are equal on the set of $\mathscr{B}$-simple functions, which is dense in $\mathfrak{L}_{\Phi}$ (condition Δ_2!). Hence, they are equal on the whole $\mathfrak{L}_{\Phi}$. This completes the first part of the proof. The converse follows immediately by virtue of Theorem 4.1.

6. Ind-additive functionals on non-gaussian random variables. Let be $S = I = [0, 1]$, $\mathscr{B}$ the σ-algebra of all Borel subsets of the unit interval and μ Lebesgue measure on it. **M** is supposed to be a symmetric homogeneous random measure on I. For the definition and properties of such a measure we refer to [12] or [14]. A random measure will be called *non-gaussian* (*gaussian*) if all its values are non-gaussian (gaussian) random variables. By [**M**] we denote the linear metric space spanned by the values of random measure **M** and closed in the F-norm induced by convergence in probability. A random variable $\xi \in [\mathbf{M}]$ if and only if it is of the form

$$\xi = \int_I \mathfrak{x}(s)\mathbf{M}(ds), \qquad \text{where} \qquad \mathfrak{x} \in \mathscr{L}(\mathbf{M}),$$

and $\mathscr{L}(\mathbf{M})$ stands for the space of all functions integrable with respect to random measure $\mathbf{M}$ (see [14]). Now, there is a one-to-one correspondence between elements of $[\mathbf{M}]$ and $\mathscr{L}(\mathbf{M})$. By introducing in $\mathscr{L}(\mathbf{M})$ the F-norm induced from $[\mathbf{M}]$ via this correspondence these two spaces become isomorphic and will be treated as equal. In [14] it was proved that to every $\mathbf{M}$ there corresponds a function $\Phi(\mathbf{M}) \in \mathscr{K}_0$ such that $\mathscr{L}(\mathbf{M}) = \mathfrak{L}_{\Phi(\mathbf{M})}$ and, conversely, for every function $\Phi \in \mathscr{K}_0$ there exists a symmetric homogeneous random measure M such that $\mathscr{L}(\mathbf{M}) = \mathfrak{L}_{\Phi}$.

A functional $\mathbf{T}$ on $[\mathbf{M}]$ is said to be *ind-additive* if it is continuous, bounded and such that the equality

$$\mathbf{T}(\xi_1 + \xi_2) = \mathbf{T}(\xi_1) + \mathbf{T}(\xi_2)$$

holds whenever random variables ξ_1 and ξ_2 are stochastically independent.

THEOREM 6.1. *For non-gaussian random measures a functional* $\mathbf{T}$ *is ind-additive on* $[\mathbf{M}]$ *if and only if* $\mathbf{T} \in \mathfrak{L}^{\oplus}_{\Phi(\mathbf{M})}$.

This theorem immediately follows from the following generalization of the Bernstein-Skitovitch-Urbanik theorem:

THEOREM 6.2. *If* $\mathbf{M}$ *is a homogeneous symmetric random measure, random variables* $\int_I \mathfrak{x}(s)\mathbf{M}(ds)$ *and* $\int_I \mathfrak{y}(s)\mathbf{M}(ds)$ *are stochastically independent and* $\mathfrak{x}(s)\mathfrak{y}(s) \neq 0$ *on the set of positive measure* μ, *then* $\mathbf{M}$ *is gaussian.*

Urbanik [13] proved this theorem under the additional assumption that $\mathbf{M}$ is a random measure with values having finite first moment. The proof in the general case is the same and we omit it.

From Theorems 5.1 and 6.1 we can easily deduce the following

COROLLARY. *If* $\mathbf{M}$ *is a non-gaussian random measure, then* $\mathbf{T}$ *is an ind-additive functional on* $[\mathbf{M}]$ *if and only if it is of the form*

$$\mathbf{T}(\xi) = \int_I K\big(\mathfrak{x}(s), s\big)\mu(ds),$$

where K *satisfies* (j)-(jjj) *of Theorem* 5.1 *and*

$$\xi = \int_I \mathfrak{x}(s)\mathbf{M}(ds).$$

As to gaussian random measures $\mathbf{N}$ with independent values, it is only known that every additive functional on $\mathfrak{L}_2 (= \mathfrak{L}_{\Phi(\mathbf{N})})$ is also ind-additive on $[\mathbf{N}]$. The converse is false by virtue of Theorem 6.2.

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