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ON AXIAL MAPS OF DIRECT PRODUCTS, II

BY

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This paper is a continuation of the joint paper [2] of Ehrenfeucht and the present author. The results of [2] were announced in [1].

Recall that a function $f\colon A_1 \times \ldots \times A_n \to A_1 \times \ldots \times A_n$ is called *axial* if there exist i and $g\colon A_1 \times \ldots \times A_n \to A_i$ such that

$$f(x_1, \ldots, x_n) = \big(x_1, \ldots, x_{i-1}, g(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n\big)$$
$$\text{for all } (x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n.$$

A function $f\colon X \to X$ which is one-to-one and onto is called a *permutation* of X.

In [2] it was proved that every permutation of $A \times B$ can be represented as a composition of five axial permutations of $A \times B$; moreover, if $|A| \neq |B|$, then four is enough; if at least one of the sets A, B is finite, then even three is enough. The first result (case $|A| \leqslant |B| = \aleph_0$) belongs to Nosarzewska (see [5]). In [1] we have asked: "Is it true that every permutation of $A \times B$ can be represented as a composition of four axial permutations of $A \times B$?" From the results of paper [2] it is clear that the question remains open only in the case $|A| = |B| \geqslant \aleph_0$. The main aim of this paper is to give the positive answer to this question (the answer was announced in [1], *Added in proof*). Our second aim is to show that in the part of [2] which deals with permutations the term "permutation" can be replaced by "one-to-one function". The basis for that extension to the case of one-to-one functions consists of theorems of [2] on permutations and of one result of the present paper. Analogous problems for arbitrary functions and functions onto were studied in [2]. Theorems (i), (ii), (iii) and (ix) of [2], strengthened in this paper, give the following results.

THEOREM 1. *Every function $f\colon A \times B \to A \times B$ which is onto can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_i: $A \times B \to A \times B$ $(i = 1, \ldots, 4)$ *are axial functions onto* (f_2, f_3, f_4 *can be permutations*).

THEOREM 2. *Every one-to-one function* f: $A \times B \to A \times B$ *can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_i: $A \times B \to A \times B$ $(i = 1, \ldots, 4)$ *are axial one-to-one functions* (f_1, f_2, f_3 *can be permutations*).

Each of these two theorems implies immediately

THEOREM 0. *Every permutation* p *of* $A \times B$ *can be represented as a composition*

$$p = p_1 \circ \ldots \circ p_4,$$

where p_i $(i = 1, \ldots, 4)$ *are axial permutations of* $A \times B$.

Let us mention that Theorems 0 and 1 are new only in the case $|A| = |B| \geqslant \aleph_0$ (see (i), (ii), (iii), (viii) and (ix) of [2]).

By a minor modification of the proofs of (iii) and (xiii) in [2], we will generalize (iii) of [2] to the following form:

THEOREM 3. *If at least one of the sets* A, B *is finite, then every one-to-one function* f: $A \times B \to A \times B$ *can be represented as a composition*

$$f = f_1 \circ f_2 \circ f_3,$$

where f_i $(i = 1, 2, 3)$ *are axial one-to-one functions* (*if* $|A| < \aleph_0$, *then* f_1 *can be of the form* $f_1(a, b) = \big(g(a, b), b\big)$ *for all* $(a, b) \in A \times B$).

It is not possible to decrease the number 4 in Theorem 0, and hence in Theorems 1 and 2 (see (v) and (v′) of [2]). Also it is not possible to decrease the number 3 in Theorem 3 (see (v″) of [2]). Some results of [2] and of this note, more precise than Theorems 0-3, are collected in Section 4. In connection with these theorems let us add that in [2] it was proved that an arbitrary function f: $A \times B \to A \times B$ can be represented as a composition of six axial functions, and if at least one of the sets A, B is infinite, then three is enough. The question of [2] "Can one decrease the number 6 in this theorem?" is still open.

In Section 5 we generalize some previous theorems to the case of functions f: $A_1 \times \ldots \times A_n \to A_1 \times \ldots \times A_n$, where $n < \omega$ (similarly as it was done in [2]), but these results are not so precise as in the case $n = 2$.

After publishing [1], I have learned from Jan Mycielski that Fred Galvin has independently obtained results which are equivalent to theorems on permutations announced in [1] (including Theorem 0 of this paper, announced in *Added in proof* in [1]). Fred Galvin has formulated these results in the more comprehensive language of groups and so do I, by his kind permission, in Section 4 of this paper.

0. Notation and terminology. In the whole paper A and B denote non-empty sets. If $|A| = |B| = \mathfrak{m} \geqslant \aleph_0$ and $f\colon A\times B \to A\times B$, then we consider two conditions concerning the function f:

$$(\mathrm{C}_1)\quad \forall(A_0\subset A)\,\forall(B_1\subset B)\Big(\big(|A_0|<\mathfrak{m}\ \&\ |B-B_1|<\mathfrak{m}\big) \Rightarrow \big(f(A\times B_1)\not\subset A_0\times B\big)\Big),$$

$$(\mathrm{C}_2)\quad \forall(B_0\subset B)\,\forall(A_1\subset A)\Big(\big(|B_0|<\mathfrak{m}\ \&\ |A-A_1|<\mathfrak{m}\big) \Rightarrow \big(f(A_1\times B)\not\subset A\times B_0\big)\Big).$$

A function $f\colon A\times B\to A\times B$ is called *vertical* if there is $g\colon A\times B\to A$ such that $f(a,b) = \big(g(a,b),b\big)$ for all $(a,b)\in A\times B$, and it is called *horizontal* if there exists $g\colon A\times B\to B$ such that $f(a,b) = \big(a, g(a,b)\big)$ for all $(a,b)\in A\times B$.

If $X = A\times B$, $a\in A$, $b\in B$, we write $X_a = \{a\}\times B$ and $X^b = A\times\{b\}$.

For every function $f\colon X\to X$, where $X = A\times B$, and for every $a\in A$, $b\in B$, we write $F_a = f(X_a)$ and $F^b = f(X^b)$.

We will often use, without explicit writing, the fact that if p is a permutation of $X = A\times B$, then the families $\{P_a\}_{a\in A}$ and $\{P^b\}_{b\in B}$ are partitions of X, and $|P_a| = |B|$ and $|P^b| = |A|$ for every $a\in A$, $b\in B$.

If p is a permutation of $X = A\times B$, then we define the function $i_p\colon X\to B$ by

$$\forall(x\in X)\,\forall(b\in B)\Big(\big(i_p(x) = b\big)\Leftrightarrow(x\in P^b)\Big).$$

Observe that if sets $Y_1, Y_2\subset X = A\times B$ are such that $i_p(Y_1)\cap i_p(Y_2) = \emptyset$, then $Y_1\cap Y_2 = \emptyset$.

A set E will be called a *selector* of a family $\{Y_u\}_{u\in U}$ if $E\subset\bigcup\{Y_u\colon u\in U\}$ and $|E\cap Y_u| = 1$ for all $u\in U$. It will be called a *partial selector* if $|E\cap Y_u|\leqslant 1$.

The following definitions will be used only in the proof of Lemma 11:

A matrix $D = (d_{a,b})_{a\in A,b\in B}$ is called a *vertical* (*horizontal*) *transformation* of a matrix $C = (c_{a,b})_{a\in A,b\in B}$ if there exists a vertical (horizontal) one-to-one function $h\colon A\times B\to A\times B$ such that $d_{a,b} = c_{h(a,b)}$ for all $(a,b)\in A\times B$. If h is a vertical (horizontal) permutation of $A\times B$, a matrix D is called a *vertical* (*horizontal*) *permutation* of a matrix C.

Let M be a matrix $\big((a,b)\big)_{a\in A,b\in B}$. For every function $f\colon A\times B\to A\times B$, we put $f(M) = \big(f(a,b)\big)_{a\in A,b\in B}$. Notice that, for any vertical (horizontal transformation N of the matrix $f(M)$, there exists a vertical (horizontal) one-to-one function $r\colon A\times B\to A\times B$ such that $N = f\circ r(M)$. If the matrix N is a permutation of the matrix $f(M)$, then r can be chosen to be a permutation of $A\times B$.

Indeed, let $N = (n_{a,b})_{a\in A,b\in B}$ be a vertical transformation of $f(M)$ and let $f(M) = (s_{a,b})_{a\in A,b\in B}$. By the definition, there exists a vertical one-to-one function $h: A\times B\to A\times B$ such that $n_{a,b} = s_{h(a,b)}$ for all $(a,b)\in A\times B$. Hence $n_{a,b} = f(h(a,b)) = f\circ h(a,b)$. Therefore $N = f\circ h(M)$. We put $r = h$. The rest is similar.

1. Permutations.

LEMMA 1. *If* $|A| = |B| = \mathfrak{m}\geqslant\aleph_0$ *and* $f\colon A\times B\to A\times B$ *is a one-to-one function or a function onto, then either* (C_1) *or* (C_2) *is satisfied.*

Proof. If neither (C_1) nor (C_2) is satisfied, then there exist sets A_0, B_0, A_1, B_1 such that $|A_0|<\mathfrak{m}$, $|B_0|<\mathfrak{m}$, $|A-A_1|<\mathfrak{m}$, $|B-B_1|<\mathfrak{m}$, $f(A\times B_1)\subset A_0\times B$ and $f(A_1\times B)\subset A\times B_0$.

Let f be a one-to-one function. Since $f(A_1\times B_1)\subset A_0\times B_0$, we have

$$|A_1\times B_1| = |f(A_1\times B_1)|\leqslant |A_0\times B_0|.$$

Hence $\mathfrak{m}<\mathfrak{m}$, a contradiction.

If f is onto, we have

$$f\big((A-A_1)\times(B-B_1)\big)\supset (A-A_0)\times(B-B_0),$$

whence

$$|(A-A_1)\times(B-B_1)|\geqslant |(A-A_0)\times(B-B_0)|.$$

But the right-hand side of the last inequality is equal to $\mathfrak{m}$, whereas the left-hand side is smaller than $\mathfrak{m}$, a contradiction.

LEMMA 2. *If* $|A| = |B| = \mathfrak{m}\geqslant\aleph_0$ *and* p *is a permutation of* $X = A\times B$ *with property* (C_1), *then there exists a set* $S\subset X = A\times B$ *with the following properties:*

(1) $$|S| = \mathfrak{m},$$

(2) $$|X_a\cap S|\leqslant 1 \quad \textit{for every } a\in A,$$

(3) $$|P^b\cap S|\leqslant 1 \quad \textit{for every } b\in B.$$

In other words, there exists a big set S (*i.e.* $|S| = \mathfrak{m}$) *which is simultaneously a partial selector of the family* $\{X_a\}_{a\in A}$ *and of the family* $\{P^b\}_{b\in B}$.

Proof. We only may assume that p is a one-to-one function, not necessarily a permutation. Let T be the set of all ordinals of power less than $\mathfrak{m}$. First we define, by transfinite induction, sequences $\{a_t\}_{t\in T}$ and $\{b_t\}_{t\in T}$ of different elements of A and B, respectively, such that $X_{a_t}\cap P^{b_t}\neq\emptyset$ for every $t\in T$. Assume that for $t'<t$ elements $a_{t'}$, $b_{t'}$ have been already defined. Put $A_0 = \{a_{t'}\colon t'<t\}$ and $B_1 = B-\{b_{t'}\colon t'<t\}$. Since p has property (C_1), we obtain $p(A\times B_1)\not\subset A_0\times B$. Hence there exists $a^0\in A-A_0$ such that $p(A\times B_1)\cap X_{a^0}\neq\emptyset$ and there exists $b^0\in B_1$ such that $P^{b^0}\cap X_{a^0}\neq\emptyset$. We put $a_t = a^0$ and $b_t = b^0$. By the definition of these two sequences, it is clear that they have the required property. Let S be a selector of the family $\{X_{a_t}\cap P^{b_t}\}_{t\in T}$. It is easy to see that S is what we need.

LEMMA 3. *Let $\{Z_w\}_{w\in W}$ be a family of pairwise disjoint sets. Let Y be a set such that*

$$|\{w\colon\ Y\cap Z_w \neq \emptyset\}| = \mathfrak{m} \geqslant \aleph_0.$$

If Y^0 is a subset of Y such that $|Y^0| < \mathfrak{m}$, then

$$|\{w\colon\ (Y-Y^0)\cap Z_w \neq \emptyset\}| = \mathfrak{m}.$$

Proof. We have

$$\{w\colon\ Y\cap Z_w \neq \emptyset\} = \{w\colon\ Y^0\cap Z_w \neq \emptyset\}\cup\{w\colon\ (Y-Y^0)\cap Z_w \neq \emptyset\}$$

and, by the disjointness of the family $\{Z_w\}_{w\in W}$,

$$|\{w\colon\ Y^0\cap Z_w \neq \emptyset\}| \leqslant |Y^0| < \mathfrak{m}.$$

Hence the lemma follows.

LEMMA 4. *If $\{Y_v\}_{v\in V}$ and $\{Z_w\}_{w\in W}$ are two families, each consisting of pairwise disjoint sets, such that*

$$(4)\qquad |\{w\colon\ Y_v\cap Z_w \neq \emptyset\}| \geqslant |V| \quad \textit{for all } v\in V,$$

then there exists a set E which is a selector of the family $\{Y_v\}_{v\in V}$ and, at the same time, a partial selector of the family $\{Z_w\}_{w\in W}$.

Proof. Assume that V is ordered in the type $|V|$. We define, by transfinite induction, a transfinite sequence $\{y_v\}_{v\in V}$ such that

$$y_v\in Y_v - \bigcup\{Z_w\colon\ \exists(v'<v)(y_{v'}\in Z_w)\} \quad \text{for every } v\in V.$$

Let $v\in V$. Assume that for $v'<v$ elements $y_{v'}$ have already been defined. Since the sets of the family $\{Z_w\}_{w\in W}$ are pairwise disjoint, we have

$$|\{w\colon\ \exists(v'<v)(y_{v'}\in Z_w)\}| < |V|.$$

Hence, by (4), there exists

$$w_0\in W - \{w\colon\ \exists(v'<v)(y_{v'}\in Z_w)\}$$

such that $Y_v\cap Z_{w_0} \neq \emptyset$. Let y_v be an arbitrary element of $Y_v\cap Z_{w_0}$. It is seen that the sequence $\{y_v\}_{v\in V}$ has the required property. Put $E = \{y_v\colon\ v\in V\}$. It is obvious that this E is what we need.

LEMMA 5. *If $|A| = |B| = \mathfrak{m} \geqslant \aleph_0$ and p is a permutation of $X = A\times B$ with property (C_1), then there exists a partition $\boldsymbol{R}$ of X with the following properties:*

$$(5)\qquad |\{a\in A\colon\ X_a\cap R \neq \emptyset\}| = \mathfrak{m} \quad \textit{for every } R\in \boldsymbol{R},$$

$$(6)\qquad |\{R\in \boldsymbol{R}\colon\ X_a\cap R \neq \emptyset\}| = \mathfrak{m} \quad \textit{for every } a\in A,$$

$$(7)\qquad |R\cap P^b| = 1 \quad \textit{for every } R\in \boldsymbol{R} \textit{ and every } b\in B.$$

Proof. Let S be a set satisfying (1), (2) and (3) for a given p. Such an S exists by Lemma 2. Let T be the set of all ordinals of power less than $\mathfrak{m}$. Let a partition $\{S_t\}_{t\in T}$ of S be such that $|S_t| = \mathfrak{m}$ for every $t \in T$. Each S_t satisfies (1), (2) and (3), since S does. Let a function $f\colon T \to A$ be such that $|f^{-1}(a)| = \mathfrak{m}$ for every $a \in A$. Put

$$B_0 = \{b\colon \sup_a |X_a \cap P^b| = \mathfrak{m}\}.$$

Let a function $g\colon B_0 \times T \to A$ be such that

$$|X_{g(b,t)} \cap P^b| > |t|+2 \quad \text{for all } (b, t) \in B_0 \times T.$$

Let C be a selector of the family

$$\{P^b \cap X_a\colon P^b \cap X_a \neq \emptyset,\ a \in A,\ b \in B\}.$$

From the definition of C it follows that

(8) $$|\{b \in B\colon P^b \cap X_a \neq \emptyset\}| = |C \cap X_a| \quad \text{for every } a \in A.$$

We show that

(9) $$|\{a\colon X_a \cap P^b \neq \emptyset\}| = \mathfrak{m} \quad \text{for every } b \in B - B_0.$$

Indeed, for every $b \in B$ we have

$$P^b = \bigcup_a (P^b \cap X_a) = \bigcup \{P^b \cap X_a\colon P^b \cap X_a \neq \emptyset,\ a \in A\}.$$

Hence for every $b \in B$ we obtain

$$\mathfrak{m} = |P^b| \leqslant |\{a\colon P^b \cap X_a \neq \emptyset\}| \sup_a |P^b \cap X_a|.$$

Let now $b \in B - B_0$. From the definition of B_0 we have

$$\sup_a |X_a \cap P^b| = \mathfrak{n}, \quad \text{where } \mathfrak{n} < \mathfrak{m}.$$

Thus $\mathfrak{m} \leqslant |\{a\colon P^b \cap X_a \neq \emptyset\}|\,\mathfrak{n}$, and hence (9) follows.

Assume that B is ordered in the type $\mathfrak{m}$.

We define, by transfinite induction, a transfinite sequence $\{R_t\}_{t\in T}$ of subsets of X such that, for every $t \in T$,

1° $R_t \cap \bigcup_{t'<t} R_{t'} = \emptyset$,

2° $D_t = \{b\colon (f(t), b) \notin \bigcup_{t'<t} R_{t'}\} \neq \emptyset$ & $(f(i), \min D_t) \in R_t$,

3° $|R_t \cap P^b| = 1$ for every $b \in B$,

4° $|R_t \cap S| = \mathfrak{m}$,

5° $\left|R_t \cap \bigcup_{T \ni t' \neq t} S_{t'}\right| \leqslant 1$,

6° $|X_a \cap R_t \cap C| \leqslant 3$ for every $a \in A$.

Let $u \in T$. Assume that for $t < u$ the sets R_t have already been defined in a way such that for every $t < u$ conditions 1°-6° are satisfied. The definition of R_u will be preceded by four observations (10)-(13).

First we prove that

$$(10) \qquad X_a - \bigcup_{t<u} R_t \neq \emptyset \quad \text{for every } a \in A.$$

Let $a \in A$.

Case 1. $|C \cap X_a| = \mathfrak{n}$, where $\mathfrak{n} < \mathfrak{m}$. Since

$$X_a = \bigcup_b X_a \cap P^b,$$

we have

$$X_a \cap R_t \subset \bigcup \{P^b \cap R_t \colon X_a \cap P^b \neq \emptyset, b \in B\} \quad \text{for every } t < u.$$

Thus

$$|X_a \cap R_t| \leqslant |\{b\colon X_a \cap P^b \neq \emptyset\}| \sup_b |P^b \cap R_t| \quad \text{for every } t < u.$$

Hence, by (8) and 3°, we have $|X_a \cap R_t| \leqslant |C \cap X_a|$ for every $t < u$. Thus, in this case, $|X_a \cap R_t| \leqslant \mathfrak{n}$ for every $t < u$. Hence

$$\Big|X_a \cap \bigcup_{t<u} R_t\Big| \leqslant \mathfrak{n}\,|u| < \mathfrak{m}.$$

Thus we have (recall that $|X_a| = |B| = \mathfrak{m}$)

$$X_a - \bigcup_{t<u} R_t \neq \emptyset.$$

Case 2. $|C \cap X_a| = \mathfrak{m}$. By 6°, we have

$$\Big|C \cap X_a \cap \bigcup_{t<u} R_t\Big| \leqslant 3\,|u| < \mathfrak{m}.$$

Hence, in this case,

$$\Big|C \cap X_a - \bigcup_{t<u} R_t\Big| = \mathfrak{m},$$

which obviously implies

$$X_a - \bigcup_{t<u} R_t \neq \emptyset.$$

Now, by 3° and property (3) of S,

$$(11) \qquad \Big|P^b \cap \Big(\bigcup_{t<u} R_t \cup S\Big)\Big| < \mathfrak{m} \quad \text{for every } b \in B.$$

Fix $b \in B - B_0$. In Lemma 3 put

$$\{Z_w\}_{w \in W} = \{X_a\}_{a \in A}, \qquad Y = P^b, \qquad Y^0 = P^b \cap \Big(\bigcup_{t<u} R_t \cup S\Big).$$

By virtue of (9) and (11) we infer that assumptions of Lemma 3 are satisfied, and thus

$$(12) \qquad \Big|\Big\{a\colon \Big(P^b - \Big(\bigcup_{t<u} R_t \cup S\Big)\Big) \cap X_a \neq \emptyset\Big\}\Big| = \mathfrak{m} \quad \text{for every } b \in B - B_0.$$

Now we observe that

$$(13)\qquad (X_{g(b,u)}\cap P^b)-\big(\bigcup_{t<u}R_t\cup C\cup S\big)\neq\emptyset \quad\text{for every } b\in B_0.$$

Indeed, for every $b\in B_0$, we have

$$|X_{g(b,u)}\cap P^b|>|u|+2 \quad \text{(by the definition of the function } g),$$

$$\big|P^b\cap\bigcup_{t<u}R_t\big|\leqslant|u| \quad (\text{by } 3^\circ),$$

$$|X_{g(b,u)}\cap P^b\cap C|\leqslant 1 \quad \text{(by the definition of the set } C),$$

$$|P^b\cap S|\leqslant 1 \quad \text{(by property (3) of the set } S),$$

whence (13) follows.

In order to define R_u, we first define some sets R_u^1, R_u^2, and then we put $R_u=R_u^1\cup R_u^2$. In view of (10) we infer that

$$D_u=\big\{b\colon \big(f(u),b\big)\notin\bigcup_{t<u}R_t\big\}\neq\emptyset.$$

Construction of R_u. Put

$$d_u=\min D_u,\qquad b_u=i_p\Big(\big(f(u),d_u\big)\Big)\qquad\text{and}\qquad S_u^*=S_u-\big(\bigcup_{t<u}R_t\cup P^{b_u}\big).$$

Observe that $|S_u^*|=\mathfrak{m}$, as a consequence of the following relations:

$$\big|S_u\cap\bigcup_{t<u}R_t\big|\leqslant|u|\cdot 1<\mathfrak{m}\quad(\text{by } 5^\circ),$$

$$|S_u\cap P^{b_u}|\leqslant 1\quad\text{(by property (3) of } S_u),$$

$$|S_u|=\mathfrak{m}\quad\text{(by property (1) of } S_u).$$

In Lemma 4 put

$$\{Y_v\}_{v\in V}=\big\{P^b-\big(\bigcup_{t<u}R_t\cup S\big)\big\}_{b\in B-(B_0\cup\{b_u\}\cup i_p(S_u^*))}\qquad\text{and}\qquad\{Z_w\}_{w\in W}=\{X_a\}_{a\in A}.$$

By (12), the assumption of Lemma 4 is satisfied in this case. Therefore, there exists a set E (it depends on u, and we shall write E_u instead of E) which is a selector of the family

$$\big\{P^b-\big(\bigcup_{t<u}R_t\cup S\big)\big\}_{b\in B-(B_0\cup\{b_u\}\cup i_p(S_u^*))}$$

and, at the same time, a partial selector of the family $\{X_a\}_{a\in A}$. We put $R_u^1=\big(f(u),d_u\big)\cup S_u^*\cup E_u$. Using the definition of R_u^1, it is easy to check that $i_p(R_u^1)\supset B-B_0$. Hence $B-i_p(R_u^1)\subset B_0$. Therefore, by (13), there exists a set R_u^2 which is a selector of the family

$$\big\{(X_{g(b,u)}\cap P^b)-\big(\bigcup_{t<u}R_t\cup C\cup S\big)\big\}_{b\in B-i_p(R_u^1)}.$$

Now we put $R_u=R_u^1\cup R_u^2$.

We have to show that conditions 1°-6° are satisfied for $t = u$. By the definition of R_u it is easy to see that conditions 1°, 2° and 3° are satisfied. To prove 4°, observe that by the definition of R_u we have $S_u^* \subset R_u$. We have proved that $|S_u^*| = \mathfrak{m}$. Since $S_u^* \subset S_u \subset S$, 4° is satisfied. By the definition of R_u^2, we have $R_u^2 \cap S = \emptyset$, and, by the definition of R_u^1,

$$R_u^1 \cap S \subset S_u^* \cup \{(f(u), d_u)\}.$$

Since $R_u = R_u^1 \cup R_u^2$, $S_u^* \subset S_u$, and $\{S_t\}_{t \in T}$ is a partition of S, we have

$$R_u \cap \bigcup_{T \ni t \neq u} S_t \subset \{(f(u), d_u)\}.$$

Hence 5° is satisfied. To prove 6°, observe that, by property (2) of S_u and by the definition of E_u, each of the three sets, the union of which is R_u^1, is a partial selector of the family $\{X_a\}_{a \in A}$. We also have

$$R_u^2 \cap C = \emptyset \quad \text{and} \quad X_a \cap R_u \cap C = X_a \cap R_u^1 \cap C \subset X_a \cap R_u^1.$$

Hence 6° is satisfied.

The defined sequence $\{R_t\}_{t \in T}$ has, in particular, properties 1°-4°. To prove Lemma 5, put $\boldsymbol{R} = \{R_t\colon t \in T\}$. We must show that $\boldsymbol{R}$ is a partition of X, and satisfies (5)-(7). To prove that $\boldsymbol{R}$ is a partition of X, by 1°, we need only to show that

$$\bigcup_{t \in T} R_t \supset X.$$

Let $(a_0, b_0) \in X$. Suppose, *a contrario*,

$$(a_0, b_0) \notin \bigcup_{t \in T} R_t.$$

Put $T_0 = \{t \in T\colon f(t) = a_0\}$. By the definition of f, $|T_0| = \mathfrak{m}$. Recall that

$$D_t = \{b\colon (f(t), b) \notin \bigcup_{t' < t} R_{t'}\} \neq \emptyset \quad \text{and} \quad (f(t), d_t) \in R_t,$$

where $d_t = \min D_t$ (by 2°). Hence, for every $t \in T_0$, we have $(a_0, d_t) \in R_t$. Therefore, by 1° we have $d_t \neq d_{t'}$ for every $t \neq t'$, $t, t' \in T_0$. Hence $|\{d_t\colon t \in T_0\}| = |T_0| = \mathfrak{m}$. But, on the other hand, since we assumed

$$(a_0, b_0) \notin \bigcup_{t \in T} R_t,$$

we have $b_0 \in D_t$ for every $t \in T_0$ (see the definition of D_t and T_0). Therefore, since $d_t = \min D_t$, we have $d_t \leqslant b_0$ for every $t \in T_0$. Hence $|\{d_t\colon t \in T_0\}| < \mathfrak{m}$. Since we have showed that $|\{d_t\colon t \in T_0\}| = \mathfrak{m}$, we have a contradiction. Thus $\boldsymbol{R}$ is a partition.

Now, by 4° and (2), property (5) is satisfied. By the definition of f we have $|f^{-1}(a)| = \mathfrak{m}$ for every $a \in A$, whence, by 2°, we obtain (6). Property (7) follows from 3°.

Remark 1. Actually, the proofs of Lemmas 2 and 5 yield the following assertion:

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two partitions of a set Y into sets such that

$$(14)\qquad |\boldsymbol{U}| = |\boldsymbol{V}| = |Y| = \mathfrak{m} \geqslant \aleph_0 \quad \text{and} \quad |U| = |V| = \mathfrak{m} \quad \text{for every } U \in \boldsymbol{U} \text{ and every } V \in \boldsymbol{V},$$

$$(15)\qquad \forall(\boldsymbol{U}' \subset \boldsymbol{U})\forall(\boldsymbol{V}' \subset \boldsymbol{V})\big[(|\boldsymbol{U}'| < \mathfrak{m} \;\&\; |\boldsymbol{V}-\boldsymbol{V}'| < \mathfrak{m}) \Rightarrow \big(\bigcup \boldsymbol{V}' \not\subset \bigcup \boldsymbol{U}'\big)\big].$$

Then there exists a partition $\boldsymbol{R}$ of Y such that each $R \in \boldsymbol{R}$ is a selector of $\boldsymbol{V}$ and

$$|\{U \in \boldsymbol{U}\colon\; U \cap R \neq \emptyset\}| = |\{R \in \boldsymbol{R}\colon\; U \cap R \neq \emptyset\}| = \mathfrak{m} \quad \text{for every } R \in \boldsymbol{R} \text{ and every } U \in \boldsymbol{U}.$$

Let us add that it is easy to check, with help for instance of Lemma 2, that if $\boldsymbol{U}$ and $\boldsymbol{V}$ are two partitions of Y which satisfy (14), then (15) is equivalent to the following:

$$(15')\qquad \forall(\boldsymbol{U}' \subset \boldsymbol{U})\forall(\boldsymbol{V}' \subset \boldsymbol{V})\big[(|\boldsymbol{V}'| < \mathfrak{m} \;\&\; |\boldsymbol{U}-\boldsymbol{U}'| < \mathfrak{m}) \Rightarrow \big(\bigcup \boldsymbol{U}' \not\subset \bigcup \boldsymbol{V}'\big)\big].$$

LEMMA 6. *If $|A| = |B| = \mathfrak{m} \geqslant \aleph_0$ and $\boldsymbol{R}$ is a partition of $X = A \times B$ with properties* (5) *and* (6), *then there exists a horizontal permutation q of X such that*

$$(16)\qquad |R \cap Q^b| = 1 \quad \textit{for every } R \in \boldsymbol{R} \textit{ and every } b \in B.$$

To prove this lemma we need three sublemmas.

If $\boldsymbol{U}$ and $\boldsymbol{V}$ are two partitions of a set Y, we may consider the condition

$$(17)\qquad |\{U \in \boldsymbol{U}\colon\; U \cap V \neq \emptyset\}| = |\{V \in \boldsymbol{V}\colon\; U \cap V \neq \emptyset\}| = |Y| = \mathfrak{m} \geqslant \aleph_0 \quad \text{for every } U \in \boldsymbol{U} \text{ and every } V \in \boldsymbol{V}.$$

SUBLEMMA 1. *Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two partitions of Y with property* (17). *Let Y_0 be a subset of Y such that $|Y_0 \cap Z| < \mathfrak{m}$ for every $Z \in (\boldsymbol{U} \cup \boldsymbol{V})$. Then partitions $\{U - Y_0\}_{U \in \boldsymbol{U}}$ and $\{V - Y_0\}_{V \in \boldsymbol{V}}$, both of $Y - Y_0$, also have property* (17).

The sublemma is evident.

SUBLEMMA 2. *Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two partitions of Y with property* (17). *For each $y_0 \in Y$ there exists a set $S \subset Y$ which is a selector of $\boldsymbol{U} \cup \boldsymbol{V}$ and $y_0 \in S$.*

Proof. Let T be the set of all ordinals of power less than $\mathfrak{m}$. Let a sequence $\{U_t\}_{t \in T}$ of sets of $\boldsymbol{U}$ be such that each $U \in \boldsymbol{U}$ appears exactly once in this sequence. Let $\{V_t\}_{t\in T}$ be a similar sequence for $\boldsymbol{V}$. We may assume that $y_0 \in U_0 \cap V_0$. We define, by transfinite induction, a sequence $\{s_t\}_{t\in T}$ of elements of Y such that $s_0 = y_0$ and, for every $t \in T$,

$$s_t \in U_t \cap V_{j(t)},$$

where

$$j(t) = \min\big(\{t'\colon\ U_t \cap V_{t'} \neq \emptyset\} - \{t'\colon\ \exists (r < t)(s_r \in V_{t'})\}\big).$$

It is easy to check that $S = \{s_t\colon\ t \in T\}$ satisfies the thesis.

SUBLEMMA 3. *Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two partitions of Y with property* (17). *Then there exists a partition $\boldsymbol{S}$ of Y such that each set $S \in \boldsymbol{S}$ is a selector of $\boldsymbol{U} \cup \boldsymbol{V}$.*

Proof. Let T be the set of all ordinals of power less than $\mathfrak{m}$. Let $\{y_t\}_{t\in T}$ be a sequence of elements of Y such that each $y \in Y$ appears exactly once in this sequence. We define, by transfinite induction, a sequence $\{S_t\}_{t\in T}$ of subsets of Y such that, for every $t \in T$,

$$S_t \cap \bigcup_{t'<t} S_{t'} = \emptyset, \qquad y_t \in \bigcup_{t'\leqslant t} S_{t'}, \qquad \text{and } S_t \text{ is a selector of } \boldsymbol{U} \cup \boldsymbol{V}. \tag{18}$$

Let $u \in T$. Assume that sets S_t for $t < u$ have already been defined in a way such that (18) is satisfied for $t < u$. In Sublemma 1 put

$$Y_0 = \bigcup_{t<u} S_t, \qquad Y = Y, \qquad \boldsymbol{U} = \boldsymbol{U} \quad \text{and} \quad \boldsymbol{V} = \boldsymbol{V}.$$

By (18) the assumptions of Sublemma 1 are satisfied and thus also the conclusion. Hence the assumptions of Sublemma 2 are satisfied for partitions $\{U - Y_0\}_{U\in\boldsymbol{U}}$ and $\{V - Y_0\}_{V\in\boldsymbol{V}}$, both of $Y - Y_0$. Therefore, by Sublemma 2, there exists a set $S \subset Y - Y_0$ which is a selector of the family $(\{U - Y_0\}_{U\in\boldsymbol{U}} \cup \{V - Y_0\}_{V\in\boldsymbol{V}})$. In the case

$$y_u \notin \bigcup_{t<u} S_t,$$

we can additionally demand that $y_u \in S$ (see Sublemma 2). We put $S_u = S$. It follows immediately from (18) that the family $\boldsymbol{S} = \{S_t\colon\ t \in T\}$ is what we need.

Proof of Lemma 6. In Sublemma 3 put $\boldsymbol{U} = \{X_a\}_{a\in A}$, $\boldsymbol{V} = \boldsymbol{R}$ and $Y = X$. In view of the assumed properties (5) and (6) of $\boldsymbol{R}$ we see that in this case the assumptions of Sublemma 3 are satisfied. Therefore, by Sublemma 3, there exists a partition $\boldsymbol{S}$ of X such that each set $S \in \boldsymbol{S}$ is a selector of $(\{X_a\}_{a\in A} \cup \boldsymbol{R})$. Thus each set $S \in \boldsymbol{S}$ has exactly one element in common with each row of the matrix $M = \big((a, b)\big)_{a\in A, b\in B}$. Hence there exists a horizontal permutation q of X such that sets $S \in \boldsymbol{S}$ are columns of $q(M)$. Therefore, each column of $q(M)$ is a selector of $\boldsymbol{R}$, since each set $S \in \boldsymbol{S}$ is. Hence the defined horizontal permutation q of X satisfies (16).

We need the following particular case of Lemma (xiii) in [2]:

LEMMA 7. *If p and q are permutations of $X = A \times B$ and there exists a partition $\boldsymbol{R}$ of X such that*

$$|Q^b \cap R| = |P^b \cap R| = 1 \quad \textit{for every } b \in B \textit{ and every } R \in \boldsymbol{R},$$

then there exist axial permutations r_1, r_2, r_3 of X such that $p = q \circ r_1 \circ r_2 \circ r_3$.

PROPOSITION I. *Let $|A| = |B| = \mathfrak{m} \geqslant \aleph_0$. If a permutation p of $X = A \times B$ has property (C_1), then it can be represented as a composition*

$$p = p_1 \circ \ldots \circ p_4,$$

where p_i $(i = 1, \ldots, 4)$ are axial permutations of X, and p_1 is horizontal.

Proof. Let p be a permutation of $X = A \times B$ with property (C_1). By Lemma 5, there exists a partition $\boldsymbol{R}$ of X with properties (5), (6) and (7). In view of (5) and (6) there exists, by Lemma 6, a horizontal permutation q of X such that (16) is satisfied for q and $\boldsymbol{R}$. Since the permutations p, q and the partition $\boldsymbol{R}$ satisfy (7) and (16), we can use Lemma 7; thus there exist axial permutations r_1, r_2, r_3 of X with $p = q \circ r_1 \circ r_2 \circ r_3$. We put $p_1 = q$, $p_2 = r_1$, $p_3 = r_2$ and $p_4 = r_3$. Hence $p = p_1 \circ \ldots \circ p_4$ and p_1 is horizontal.

It is easy to see that Proposition I is equivalent to

PROPOSITION I′. *If $|A| = |B| = \mathfrak{m} \geqslant \aleph_0$, then every permutation p of $X = A \times B$ with property (C_2) can be represented as a composition*

$$p = p_1 \circ \ldots \circ p_4,$$

where p_i $(i = 1, \ldots, 4)$ are axial permutations, and p_1 is vertical.

Now we can strengthen Theorem (i) of [2] to Theorem 0. Indeed, if $|A| = |B| \geqslant \aleph_0$, then Theorem 0 follows from Lemma 1 and Propositions I and I′. In the case of other cardinalities this theorem follows from Theorems (ii) and (iii) of [2].

Remark 2. One can prove that for each limit cardinal $\mathfrak{m}$ the converse of Proposition I holds true. It does not hold for any cardinal $\mathfrak{m}$ which is not limit.

2. Functions onto. We need the following particular case of Lemma (viii) of [2]:

LEMMA 8. *If $|A| \leqslant |B|$, then every function $f\colon A \times B \to A \times B$ which is onto can be represented as a composition $f = g \circ p$, where $g\colon A \times B \to A \times B$ is a horizontal function onto, and p is a permutation of $A \times B$.*

(If $|B| < \aleph_0$, then the lemma is obvious, since f must be a permutation.)

PROPOSITION II. *If $|A| = |B| = \mathfrak{m} \geqslant \aleph_0$, then every function $f\colon A \times B \to A \times B$ which is onto and has property (C_1) can be represented as* *a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_1: $A \times B \to A \times B$ *is a horizontal function onto, and* f_2, f_3, f_4 *are axial permutations of* $A \times B$.

Proof. Let $f = g \circ p$, where g and p are as in Lemma 8. We show that p has property (C_1). For otherwise there would exist sets $A_0 \subset A$ and $B_1 \subset B$ such that $|A_0| < \mathfrak{m}$, $|B - B_1| < \mathfrak{m}$ and $p(A \times B_1) \subset A_0 \times B$, and since g is horizontal, we would have

$$g \circ p(A \times B_1) \subset g(A_0 \times B) \subset A_0 \times B.$$

Hence $f(A \times B_1) \subset A_0 \times B$, contrary to the assumption that f has property (C_1). By Proposition I, there exist axial permutations p_1, p_2, p_3, p_4 such that $p = p_1 \circ \ldots \circ p_4$, and p_1 is horizontal. Thus we have

$$f = g \circ p_1 \circ p_2 \circ p_3 \circ p_4,$$

where, in particular, g and p_1 are horizontal functions onto. Put $f_1 = g \circ p_1$, $f_2 = p_2, f_3 = p_3, f_4 = p_4$. Therefore, $f = f_1 \circ \ldots \circ f_4$, where f_1 is a horizontal function onto, and f_2, f_3, f_4 are axial permutations.

It is easy to see that Proposition II is equivalent to the following

PROPOSITION II′. *If* $|A| = |B| = \mathfrak{m} \geqslant \aleph_0$, *then every function* f: $A \times B \to A \times B$ *which is onto and has property* (C_2) *can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_1: $A \times B \to A \times B$ *is a vertical function onto, and* f_2, f_3, f_4 *are axial permutations of* $A \times B$.

Now we can strengthen one of the results of [2] about functions onto to Theorem 1. Indeed, in the case $|A| = |B| \geqslant \aleph_0$ Theorem 1 follows from Lemma 1 and Propositions II and II′. In the remaining case Theorem 1 was proved in [2] (if at least one of the sets A, B is finite, then Theorem 1 is a consequence of a particular case of (ix) of [2], and if $\aleph_0 \leqslant |A| \neq |B| \geqslant \aleph_0$, one can use simultaneously (ii) and (viii) of [2]).

3. One-to-one functions.

LEMMA 9. *If* B *is infinite (while* A *may be of arbitrary finite or infinite cardinality), then every one-to-one function* f: $A \times B \to A \times B$ *can be represented as a composition* $f = p \circ g$, *where* p *is a permutation of* $A \times B$ *and* g: $A \times B \to A \times B$ *is a horizontal one-to-one function.*

SUBLEMMA 4. *There exists a set* $S \subset X = A \times B$ *such that* $|S| = |X - f(X)|$ *and* $|X_a - S| = |B|$ *for every* $a \in A$.

Proof. Case $|A| \leqslant |B| \geqslant \aleph_0$. In this case we have $|X| = |B|$. Let $B_0 \subset B$ be a set such that $|B_0| = |X - f(X)|$ and $|B - B_0| = |B|$. Let $a_0 \in A$. We put $S = \{a_0\} \times B_0$.

Case $|A| \geqslant |B| \geqslant \aleph_0$. Here we have $|X| = |A|$. Let a set $A_0 \subset A$ be such that $|A_0| = |X - f(X)|$. Let $b_0 \in B$. We put $S = A_0 \times \{b_0\}$.

Proof of Lemma 9. Let S be a set as in Sublemma 4. Let a function $g\colon X \to X$ be one-to-one and horizontal and such that $g(X) = X - S$ (such a g exists by the property of S). Let a function $h\colon X \to X$ be one-to-one and such that $h(S) = X - f(X)$. We put

$$p(x) = \begin{cases} f(g^{-1}(x)) & \text{if } x \in g(X), \\ h(x) & \text{if } x \in X - g(X). \end{cases}$$

By the definition of p we conclude that p is a permutation of X and $f = p \circ g$.

PROPOSITION III. *If $|A| \neq |B| \geqslant \aleph_0$, then every one-to-one function $f\colon A \times B \to A \times B$ can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_1, f_2, f_3 are axial permutations, f_4 is an axial one-to-one function, and f_1 is vertical.

Proof. Case $|A| < \aleph_0$. In this case the proposition follows from (iii) of [2] and from Lemma 9.

Case $|A| \geqslant \aleph_0$. By Lemma 9 we have $f = p \circ g$, where p is a permutation of $A \times B$ and $g\colon A \times B \to A \times B$ is a horizontal one-to-one function. By (ii) of [2] we have $p = p_1 \circ \ldots \circ p_4$, where p_i $(i = 1, \ldots, 4)$ are axial permutations, and p_1 is vertical. Hence we may assume, without loss of generality, that p_4 is horizontal. Put $f_1 = p_1$, $f_2 = p_2$, $f_3 = p_3$, $f_4 = p_4 \circ g$. We have $f = f_1 \circ \ldots \circ f_4$. The additional claim about $f_1, \ldots, f_4$ is also visible from their definitions.

It is easy to see that Proposition III is equivalent to the following

PROPOSITION III'. *If $\aleph_0 \leqslant |A| \neq |B|$, then every one-to-one function $f\colon A \times B \to A \times B$ can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_1, f_2, f_3 are axial permutations, f_4 is an axial one-to-one function, and f_1 is horizontal.

PROPOSITION IV. *If $|A| = |B| \geqslant \aleph_0$, then every one-to-one function $f\colon A \times B \to A \times B$ with property (C_1) can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_1, f_2, f_3 are axial permutations, f_4 is an axial one-to-one function, and f_1 is horizontal.

Proof. Since both A, B are infinite, using Lemma 9 we can represent f as $p \circ g$, where p is a permutation of $A \times B$ and $g\colon A \times B \to A \times B$ is a vertical one-to-one function. Now, p has property (C_1). For if not, then there would exist sets $A_0 \subset A$ and $B_1 \subset B$ such that $|A_0| < \mathfrak{m}$, $|B - B_1| < \mathfrak{m}$ and $p(A \times B_1) \subset A_0 \times B$. Since g is vertical, we would have $g(A \times B_1) \subset A \times B_1$ and, therefore,

$$p \circ g(A \times B_1) \subset A_0 \times B.$$

Hence $f(A \times B_1) \subset A_0 \times B$, contrary to the assumption that f has property (C_1).

Since we have proved that p has property (C_1), by Proposition I we can represent p as $p_1 \circ p_2 \circ p_3 \circ p_4$, where p_i $(i = 1, 2, 3, 4)$ are axial permutations, and p_1 is horizontal. Hence we may assume, without loss of generality, that p_4 is vertical. Put $f_i = p_i$ for $i = 1, 2, 3$, and $f_4 = p_4 \circ g$. We have $f = f_1 \circ \ldots \circ f_4$. The additional claim about $f_1, \ldots, f_4$ is also visible from their definitions.

It is easy to see that Proposition IV is equivalent to the following

PROPOSITION IV′. *If $|A| = |B| \geqslant \aleph_0$, then every one-to-one function $f\colon A \times B \to A \times B$ with property (C_2) can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_4,$$

where f_1, f_2, f_3 are axial permutations, f_4 is an axial one-to-one function, and f_1 is vertical.

Now we can generalize Theorem 0 to the case of one-to-one functions, namely to Theorem 2.

Indeed, in the case $|A| = |B| \geqslant \aleph_0$ Theorem 2 follows from Propositions IV, IV′ and Lemma 1. In the case $|A| \neq |B| \geqslant \aleph_0$ or $\aleph_0 \leqslant |A| \neq |B|$ the theorem follows from Propositions III and III′. In the case $|A| < \aleph_0$ and $|B| < \aleph_0$, f must be a permutation and the theorem follows from (iii) of [2].

We need the following known fact:

LEMMA 10. *If a is a positive integer, $\boldsymbol{P}$ is a partition of a set X into a-element sets, and $\boldsymbol{Q}$ is a family of pairwise disjoint a-element subsets of X, then there exists a partition $\boldsymbol{R}$ of X such that $|R \cap Y| = 1$ for every $R \in \boldsymbol{R}$ and every $Y \in \boldsymbol{P} \cup \boldsymbol{Q}$.*

We now show that Lemma 10 follows from Lemma (xi) of [2] (which, in its turn, is an almost immediate consequence of Theorem 10.1.5 in [4]). Let X_1 be a set such that $X_1 \cap X = \emptyset$ and $|X_1| \geqslant \aleph_0$. It is easy to define two partitions $\boldsymbol{P}^*$ and $\boldsymbol{Q}^*$ of $Z = X_1 \cup X$ into a-element sets such that $\boldsymbol{P}$ is a subfamily of $\boldsymbol{P}^*$, and $\boldsymbol{Q}$ is a subfamily of $\boldsymbol{Q}^*$. By Lemma (xi) of [2] applied to Z, $\boldsymbol{P}^*$ and $\boldsymbol{Q}^*$, we infer that there exists a partition $\boldsymbol{R}^*$ of Z such that $|S \cap Y| = 1$ for every $S \in \boldsymbol{R}^*$ and every $Y \in \boldsymbol{P}^* \cup \boldsymbol{Q}^*$. Put

$$\boldsymbol{R} = \boldsymbol{R}^* \cap X = \{S \cap X\colon S \in \boldsymbol{R}^*\}.$$

From the definitions of $\boldsymbol{R}$ and $\boldsymbol{R}^*$ it is easily seen that $\boldsymbol{R}$ is what we need.

Now we modify a part of Lemma (xiii) of [2]:

LEMMA 11. *Let a function* $f\colon A\times B\to A\times B$ *be one-to-one and let* $\boldsymbol{P}=\{A\times\{b\}\colon b\in B\}$ *and* $\boldsymbol{Q}=\{f(A\times\{b\})\colon b\in B\}$. *Suppose that there exists a partition* $\boldsymbol{R}$ *of* $X=A\times B$ *with the property*

$$|R\cap Y|=1 \quad \textit{for every } R\in\boldsymbol{R} \textit{ and every } Y\in\boldsymbol{P}\cup\boldsymbol{Q}.$$

Then there exist axial one-to-one functions f_1, f_2, f_3 *with* $f=f_1\circ f_2\circ f_3$, *and* f_1 *is vertical.*

Proof. Let M be the matrix $((a,b))_{a\in A,b\in B}$. Note that if $h\colon X\to X$ is a one-to-one function, then the matrix $h(M)$ consists of different elements. By the assumption of the lemma, $\boldsymbol{P}$ is the set of columns of M, and $\boldsymbol{Q}$ is the set of columns of $f(M)$. Thus each set $R\in\boldsymbol{R}$ has exactly one element in common with each column of M and exactly one element in common with each column of $f(M)$. Therefore, there exists a vertical permutation f_1 of X such that

(a) the sets $R\in\boldsymbol{R}$ are the rows of the matrix $f_1(M)$.

Put $\boldsymbol{R}^*=\{R^*\colon R\in\boldsymbol{R}\}$, where $R^*=R\cap f(X)$ for every $R\in\boldsymbol{R}$. It is easy to see that the family $\boldsymbol{R}^*$ has the following properties:

(b) $\boldsymbol{R}^*$ is a partition of $f(X)$,

(c) $|R^*|=|R|=B$ for every $R\in\boldsymbol{R}$,

(d) $R^*\subset R$ for every $R\in\boldsymbol{R}$,

(e) for every $R\in\boldsymbol{R}$, R^* has exactly one element in common with each column of $f(M)$.

By virtue of (a), (c) and (d), there exists a horizontal transformation of the matrix $f_1(M)$ and hence a horizontal one-to-one function $g\colon X\to X$ such that the sets $Y\in\boldsymbol{R}^*$ are the rows of $f_1\circ g(M)$. Therefore, by (b) and (e), there exists a horizontal permutation of the matrix $f_1\circ g(M)$ and hence a horizontal permutation g' of X such that the set of elements of each column of $f_1\circ g\circ g'(M)$ coincides with the set of elements of the column of $f(M)$ with the same index. Thus there exists a vertical permutation of $f_1\circ g\circ g'(M)$ and hence a vertical permutation f_3 of X such that $f_1\circ g\circ g'\circ f_3(M)=f(M)$. Put $f_2=g\circ g'$. We have $f=f_1\circ f_2\circ f_3$, where $_1, f_2, f_3$ are such as we need.

Now we can generalize Theorem (iii) of [2] to Theorem 3. (Notice that since $|A|<\aleph_0$, f_1 and f_3 in Theorem 3 must be permutations.) For this purpose put

$$\boldsymbol{P}=\{A\times\{b\}\colon b\in B\} \quad\text{and}\quad \boldsymbol{Q}=\{f(A\times\{b\})\colon b\in B\},$$

and apply Lemmas 10 and 11.

4. Recapitulation of some known facts about axial functions.

Remark 3. *If* $2 \leqslant |A| \leqslant |B| \geqslant \aleph_0$, *then there exists a function* f: $A \times B \to A \times B$ *which is onto and such that*

$$f \neq f_1 \circ \ldots \circ f_4,$$

whenever f_i: $A \times B \to A \times B$ $(i = 1, \ldots, 4)$ *are axial functions onto, and* f_1 *is vertical.*

Proof. Put $X = A \times B$. Let $(a_0, b_0) \in X$. Let a function f: $X \to X$ be such that $f(X - X_{a_0}) = \{(a_0, b_0)\}$ and $f(X_{a_0}) = X - \{(a_0, b_0)\}$. We have to show that f satisfies Remark 3. Suppose to the contrary that there exist axial functions f_1, f_2, f_3, f_4 onto and such that $f = f_1 \circ \ldots \circ f_4$ and f_1 is vertical. We may assume, without loss of generality, that f_3 is vertical and that f_2, f_4 are horizontal. Therefore, $f_1(X^b) = f_3(X^b) = X^b$ for every $b \in B$, and $f_2(X_a) = f_4(X_a) = X_a$ for every $a \in A$, since f_1, f_2, f_3, f_4 are also onto. Hence we have

$$f_1 \circ f_2 \circ f_3(X - X_{a_0}) = f_1 \circ f_2 \circ f_3 \circ f_4(X - X_{a_0}) = f(X - X_{a_0}) = \{(a_0, b_0)\},$$
$$f_1 \circ f_2 \circ f_3(X_{a_0}) = f_1 \circ f_2 \circ f_3 \circ f_4(X_{a_0}) = f(X_{a_0}) = X - \{(a_0, b_0)\}.$$

It follows that, for every $b \in B$,

(a) $$|f_1 \circ f_2(X^b)| = 2 \quad \text{and} \quad (a_0, b_0) \in f_1 \circ f_2(X^b).$$

Case 1. $|A| \geqslant 3$. We show that there exists $b \in B$ such that $|f_1 \circ f_2(X^b) - \{(a_0, b_0)\}| \geqslant 2$, contrary to (a). We choose elements a_1, a_2, $a_3 \in A$ one after another and $b_1 \in B$ in a way such that

$$a_1 \neq a_0, \quad (a_2, b_0) \in f_1^{-1}(a_1, b_0), \quad (a_3, b_0) \notin f_1^{-1}(\{(a_0, b_0), (a_1, b_0)\}),$$
$$f_2(a_2, b_1) = (a_2, b_0).$$

Hence we have

(b) $$f_1 \circ f_2(a_2, b_1) = (a_1, b_0).$$

Observe that

(c) $$f_2(a_3, b_1) \notin f_1^{-1}(\{(a_0, b_0), (a_1, b_0)\}).$$

For suppose to the contrary that $f_2(a_3, b_1) \in f_1^{-1}(\{(a_0, b_0), (a_1, b_0)\})$. Then $f_2(a_3, b_1) = (a_3, b_0)$, since f_1 is vertical and f_2 is horizontal. Hence

$$(a_3, b_0) \in f_1^{-1}(\{(a_0, b_0), (a_1, b_0)\})$$

which is impossible by the property of a_3.

From (c) we get immediately

(d) $$f_1 \circ f_2(a_3, b_1) \notin \{(a_0, b_0), (a_1, b_0)\}.$$

From (b) and (d) we have

(e) $$f_1 \circ f_2(a_2, b_1) \neq f_1 \circ f_2(a_3, b_1).$$

Since $a_1 \neq a_0$, we infer from (b) and (d) that

(f) $$f_1 f_2(a_2, b_1), f_1 f_2(a_3, b_1) \in f_1 f_2(X^{b_1}) - \{(a_0, b_0)\}.$$

From (e) and (f) we have $|f_1 f_2(X^{b_1}) - \{(a_0, b_0)\}| \geqslant 2$.

Case 2. $|A| < \aleph_0$. In this case f_1 and f_3 must be vertical permutations. Thus there exists $a_1 \in A$ such that $f_1^{-1}(a_0, b_0) = (a_1, b_0)$. Let $b_1 \in B$ be such that $f_2(a_1, b_1) \neq (a_1, b_0)$. We have

(g) $$f_1 \circ f_2(X^{b_1}) \subset f_1 \circ f_2(a_1, b_1) \cup f_1(X - X_{a_1})$$

in view of $X^{b_1} \subset \{(a_1, b_1)\} \cup (X - X_{a_1})$ and $f_2(X - X_{a_1}) = X - X_{a_1}$. Since $f_2(a_1, b_1) \neq (a_1, b_0) = f_1^{-1}(a_0, b_0)$, we have

(h) $$f_1 \circ f_2(a_1, b_1) \neq (a_0, b_0).$$

Further $(a_0, b_0) \in f_1(X_{a_1})$, since $f_1(a_1, b_0) = (a_0, b_0)$. Hence, since f_1 is a permutation, we have

(i) $$(a_0, b_0) \notin f_1(X - X_{a_1}).$$

From (g), (h) and (i) it follows that $(a_0, b_0) \notin f_1 \circ f_2(X^{b_1})$, contrary to (a).

Since either Case 1 or Case 2 is satisfied, the proof of Remark 3 is complete.

Now we collect some results of paper [2], a part of them in the form strengthened and generalized by results of the present paper.

Let $n < \omega$, let $A_1, \ldots, A_n$ be sets, each of cardinality greater than 1, $\pi_i \colon A_1 \times \ldots \times A_n \to A_i$ — the projection, F — the semigroup of all functions $f \colon A_1 \times \ldots \times A_n \to A_1 \times \ldots \times A_n$, S — the semigroup of all functions $f \in F$ which are onto, J — the semigroup of all functions $f \in F$ which are one-to-one, P — the group of all functions $f \in F$ which are permutations. We put, for every $i \leqslant n$,

$$F_i = \{f \in F \colon \pi_j f = \pi_j \text{ for every } j \neq i\}, \quad S_i = S \cap F_i,$$

$$J_i = J \cap F_i, \quad P_i = P \cap F_i.$$

Notice that P_i is the subgroup of P, and F_i, J_i, S_i are subsemigroups of F, J, S, respectively. It is easy to see that if $n = 2$, then F_1 is the family of all vertical functions $f \colon A_1 \times A_2 \to A_1 \times A_2$, F_2 is the family of all horizontal functions, and $F_1 \cup F_2$ is the family of all axial functions. Let $n = 2$, $|A_1| \geqslant 2$ and $|A_2| \geqslant 2$. Then we have

1° *If* $|A_1| \leqslant |A_2| < \aleph_0$, *then* $P = S = J = P_1 P_2 P_1 = P_2 P_1 P_2 \neq P_1 P_2 \cup \cup P_2 P_1$.

2° *If* $\aleph_0 > |A_1| < |A_2| \geqslant \aleph_0$, *then* $P = P_1P_2P_1 \neq P_2P_1P_2$.

3° *If* $\aleph_0 \leqslant |A_1| < |A_2|$, *then* $P = P_1P_2P_1P_2 = P_2P_1P_2P_1 \neq P_1P_2P_1 \cup \cup P_2P_1P_2$.

4° *If* $|A_1| = |A_2| \geqslant \aleph_0$, *then* $P = P_1P_2P_1P_2 \cup P_2P_1P_2P_1 \neq P_1P_2P_1P_2$.

5° *If* $|A_1| < |A_2| \geqslant \aleph_0$, *then* $S = S_2P_1P_2P_1 \neq S_1S_2S_1S_2$.

6° *If* $|A_1| = |A_2| \geqslant \aleph_0$, *then* $S = S_1P_2P_1P_2 \cup S_2P_1P_2P_1 \neq S_1S_2S_1S_2$.

7° *If* $\aleph_0 > |A_1| < |A_2| \geqslant \aleph_0$, *then* $J = P_1J_2P_1 = P_1P_2P_1J_2 \neq J_2J_1J_2$.

8° *If* $\aleph_0 \leqslant |A_1| < |A_2|$, *then* $J = P_1P_2P_1J_2 = P_2P_1P_2J_1 \neq J_1J_2J_1 \cup \cup J_2J_1J_2$.

9° *If* $|A_1| = |A_2| \geqslant \aleph_0$, *then* $J = P_1P_2P_1J_2 \cup P_2P_1P_2J_1 \neq J_1J_2J_1J_2$.

10° *If* $|A_1| \leqslant |A_2| < \aleph_0$, *then* $F = P_1P_2F_1F_2P_1P_2 = P_2P_1F_2F_1P_2P_1$.

11° *If* $|A_1| \leqslant |A_2| \geqslant \aleph_0$, *then* $F = F_2F_1F_2$.

Theorems 1°, 2°, 3°, 5°, 10° and 11° are from [2], and Theorems 1°-11° are not independent. Theorems 1°-4° were obtained independently by Fred Galvin. Theorems and lemmas of [2] have been denoted by (i), (ii), ..., (v), (v′), (v″), (vi), ..., (xiii). Remark that if $|A_1| < \aleph_0$, then $P_1 = S_1 = J_1$.

1° follows from (iii) and (v″).

2° follows from (iii); the part $\neq$ follows from Remark 3 and (viii), or with help of (xiii).

3° follows from (ii) and (v′).

4° follows from Theorem 0 and (v).

5° follows from (ii), (viii) and Remark 3.

6° follows from Theorem 1 and (v).

7° follows from Theorem 3, Proposition III and the part $\neq$ of 2°.

8° follows from Propositions III, III′ and (v′).

9° follows from Theorem 2 and (v).

10° follows from the proof of (iv).

11° follows from the proof of (vii).

5. An extension to arbitrary finite number of sets. It follows from 1°, 7°, 8° and 9° of Section 4 that, in particular:

1. *Every one-to-one function* $f\colon A_1 \times A_2 \to A_1 \times A_2$ *can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_5,$$

where $f_i\colon A_1 \times A_2 \to A_1 \times A_2$ $(i = 1, \ldots, 5)$ *are one-to-one functions,* f_1, f_3, f_5 *are vertical, and* f_2, f_4 *are horizontal.*

2. *If* $|A_1| < \aleph_0$, *then every one-to-one function* $f\colon A_1 \times A_2 \to A_1 \times A_2$ *can be represented as a composition*

$$f = f_1 \circ \ldots \circ f_3,$$

where f_i $(i = 1, 2, 3)$ *are one-to-one functions,* f_1 *and* f_3 *are vertical, and* f_2 *is horizontal.*

Let $k: \omega\times\omega \to \omega$ be the function defined in (vi) of [2]. We can generalize Statements 1 and 2 as follows:

If $A_1, \ldots, A_m$ *are finite and* $B_1, \ldots, B_n$ *are infinite, then every one-to-one function*

$$f: A_1\times \ldots \times A_m\times B_1\times \ldots \times B_n \to A_1\times \ldots \times A_m\times B_1\times \ldots \times B_n$$

can be represented as a composition

$$f = f_1\circ \ldots \circ f_{k(m,n)},$$

where all f_i *for* $i = 1, \ldots, k(m, n)$ *are axial one-to-one functions (and* $f_1(x_1, \ldots, x_{m+n}) = \big(g(x_1, \ldots, x_{m+n}), x_2, \ldots, x_{m+n}\big)$ *for every* $(x_1, \ldots, x_{m+n})$ *in* $A_1\times \ldots \times A_m\times B_1\times \ldots \times B_n$*).*

The proof is exactly the same as that of (vi) in [2]. It is clear that this theorem implies (vi) of [2].

Let us mention that we may assume in this theorem (and in (vi) and (ix) of [2]) the function $k: \omega\times\omega \to \omega$ taking for some $n < \omega$ the values less than the values of the original function from [2]. But since such a new function k has an unpleasant definition and I do not know if it is then as small as possible, I prefer the original function from [2].

The following problem is open:

P 960. To find the function k satisfying Theorem (vi) of [2] and as small as possible.

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