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DEGREES OF NON-DEFINABILITY OF THE SACKS MODEL

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In this note* we show that the Sacks notion of forcing is homogeneous and, therefore, there are two degrees of non-definability in the Sacks model [2].

We start with some topological facts.

Let P be the set of all non-empty perfect subsets of the interval $I = [0, 1]$ ordered by inclusion.

THEOREM 1. *If $F_1, F_2 \in P$, then there is a homeomorphism*

$$h: I \xrightarrow{\text{onto}} I$$

such that $h(F_1) \cap F_2 \in P$.

Observe that if F_1 or F_2 contains an interval, then we can clearly find a required homeomorphism. Thus it remains to consider the case where F_1 and F_2 are nowhere dense sets. We divide the proof into some steps. Let us state the following easy proposition:

PROPOSITION. *Let $F \in P$ be a nowhere dense set such that $0, 1 \in F$ and let $S(F)$ be the set of centres of all components of $I - F$. Then*

(i) $S(F) \cap F = 0$,

(ii) $\overline{S(F)} \supseteq F$,

(iii) *$S(F)$ is ordered by $<$ in type η.*

LEMMA 1. *If $F_1, F_2 \in P$ are nowhere dense and $0, 1 \notin (F_1 - F_2) \cup \cup (F_2 - F_1)$, then there is an order isomorphism*

$$T: \langle S(F_1), < \rangle \to \langle S(F_2), < \rangle$$

which can be extended to an order isomorphism

$$\tilde{T}: I - F_1 \to I - F_2.$$

Proof. The existence of T follows easily from the Proposition. Let $\langle I_n^j : n \in \omega \rangle$ be the enumeration of components of $I - F_j$ $(j = 1, 2)$ and

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let S_n^j be the centre of I_n^j. Now we define

$$\tilde{T} \upharpoonright I_n^1 : I_n^1 \xrightarrow{\text{onto}} I^2_{T(S_n^1)}$$

as a linear order-preserving mapping. Then $\tilde{T}$ has the required properties.

LEMMA 2. *If $F_1, F_2 \in P$ are nowhere dense sets and $0, 1 \notin (F_1 - F_2) \cup \cup (F_2 - F_1)$, then there exists a homeomorphism*

$$h\colon I \xrightarrow{\text{onto}} I$$

such that $h(F_1) = F_2$.

Proof. Take the isomorphism $\tilde{T} : I - F_1 \to I - F_2$ from Lemma 1 and put

$$h(x) = \sup\{T(s)\colon\ s \in I - F_1 \ \&\ s < x\}.$$

LEMMA 3. *If $F_1, F_2 \in P$ are nowhere dense sets, then there is a homeomorphism*

$$h : I \xrightarrow{\text{onto}} I$$

such that $h(F_1) \cap F_2 \in P$.

Proof. There is $\varepsilon > 0$ such that $F_1 \cap [\varepsilon, 1-\varepsilon]$ and $F_2 \cap [\varepsilon, 1-\varepsilon]$ are elements of P.

Thus Theorem 1 follows easily from Lemma 3.

We assume that the reader is familiar with all necessary notation concerning forcing technique as well as with the notion of degrees of non-definability.

In the sequel let $\mathscr{M}$ denote a countable standard model of ZFC and let P be a set of all non-empty perfect subsets of the interval I constructed in the model $\mathscr{M}$ and ordered by inclusion. P is called the *Sacks notion of forcing.*

THEOREM 2. *Let G be a P-generic over $\mathscr{M}$. Then in $\mathscr{M}[G]$ there are only two degrees of non-definability.*

Proof. We recall the following classical theorems:

1 (Sacks [2]). *If $x \in \mathscr{M}[G] \setminus \mathscr{M}$, then $\mathscr{M}[G] \models V = L[x]$.*

2 (Lévy [1], p. 127-151). *If C is a homogeneous notion of forcing in $\mathscr{M}$, then for every G C-generic over $\mathscr{M}$ we have $\mathscr{M}[G] \models \mathrm{HOD} = L$.*

From the theorem of Lévy we infer that, in the Sacks model, $\mathscr{M}[G] \models \mathrm{HOD} = L$. Now let $x \in \mathscr{M}[G] - L$. Then $L[x] = \mathscr{M}[G]$, but $L[x] \subseteq \subseteq \mathrm{HOD}(x)$. Hence

$$\mathscr{M}[G] \models V = \mathrm{HOD}(x).$$

REFERENCES

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[2] G. Sacks, *Forcing with perfect closed sets*, Proceedings of the Symposium in Pure Mathematics 13 (1) (1971), p. 331-356.

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