COLLOQUIUM MATHEMATICUM

VOL. XLV 1981 FASC. 2

A NOTE ON THE PAPER OF SCHÜTT "UNCONDITIONALITY IN TENSOR PRODUCTS"

BY

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In [2] Gordon and Lewis gave the first example of a sequence (E_n) of finite-dimensional Banach spaces whose unconditional basis constants (and even constants of local unconditional structure) tend to infinity with n. Their proof uses the language of Banach ideals and is rather complicated. It was recently simplified by Schütt in [3]. In this paper* we give a simple proof of the main theorem of [3]. As a corollary we observe that the space of operators on l_m^2 equipped with a unitary invariant norm has large unconditional basis constant unless this norm is close, in a sense, to the Hilbert-Schmidt norm. In particular, in the case of the usual operator norm this constant is not smaller than $m^{1/2}/8$ (on the other hand, it is obvious that it does not exceed $m^{1/2}$). Other corollaries may be found in [3].

Let us recall some definitions. Let E be a finite-dimensional Banach space over the real field (in this paper we consider only such spaces) and let $(x_k)_{k=1}^n$ be its basis. We define the *unconditional constant of the basis* (x_k) as

$$\mathrm{ubc}(x_k) = \sup \|T_\varepsilon\|,$$

where

$$T_\varepsilon\colon \sum t_k x_k \to \sum \varepsilon_k t_k x_k \tag{1}$$

and the supremum is taken over all $\varepsilon = (\varepsilon_k)_{k=1}^n \in \{-1, 1\}^n$. The *unconditional basis constant* of E is defined as follows:

$$\mathrm{ubc}(E) = \inf \mathrm{ubc}(x_k),$$

where the infimum is taken over all bases (x_k) of E.

We define also the *constant of local unconditional structure* of E by

$$\mathrm{lust}(E) = \inf \|u\|\,\|v\|\,\mathrm{ubc}(F)$$

* Supported in part by NSF Grant 76-0734.

with the infimum taken over all finite-dimensional spaces F and all pairs of linear operators $u\colon E\to F$ and $v\colon F\to E$ such that $v\circ u=\mathrm{id}_E$. Obviously, $\mathrm{lust}(E)\leqslant \mathrm{ubc}(E)$.

We say that a norm a on the space of operators on l_m^2 is *unitary invariant* if $a(T)=a(UTV)$ for any unitary operators U and V. It is well known (see, e.g., [1]) that there exists a one-to-one correspondence between such unitary invariant norms and symmetric norms on R^m (i.e. norms $\|\cdot\|$ satisfying $\|(t_k)\|=\|(\varepsilon_k t_{\pi(k)})\|$ for any choice of signs ε_k, any permutation π and any $(t_k)\in R^m$). This correspondence is induced by the formula $a(\Lambda)=\|(\lambda_k)\|$, where Λ is the diagonal operator given by the sequence (λ_k). In particular, if $\|(t_k)\|=\left(\sum|t_k|^p\right)^{1/p}$ (the l_m^p-norm), we obtain the trace classes with the unitary invariant norm c_p; for $p=2$ we get the Hilbert-Schmidt norm, and for $p=\infty$ the operator norm.

Finally, for any finite set S and any function $f\colon S\to R$, we define

$$\operatorname*{Av}_{s\in S} f(s)=(\operatorname{card}S)^{-1}\sum_{s\in S}f(s).$$

Now we are able to formulate the following

THEOREM (Schütt). *Let E be a B-space with* $\dim E=n$ *and let* (x_k) *be its basis. Assume that for some* $G\subset\{-1,1\}^n$ *and constants* K, M *the following conditions are satisfied*:

(i) $\|T_\varepsilon\|\leqslant K$ *for* $\varepsilon\in G$ (cf. (1));

(ii) $M^{-1}\left(\sum|t_k|^2\right)^{1/2}\leqslant(\operatorname{card}G)^{-1}\left|\sum t_k\varepsilon_k\right|$ *for* $(t_k)\in R^n$.

Then $\mathrm{ubc}(x_k)\leqslant K^2M^2\,\mathrm{lust}(E)$.

COROLLARY. *Let a be a unitary invariant norm on the space of operators on l_m^2. Let $\|\cdot\|$ be the corresponding symmetric norm on R^m. Set $E=(R^{m^2},a)$. Then*

$$\mathrm{lust}(E)\geqslant\tfrac{1}{8}\sup\left\{\max(\|t\|,\|t\|^{-1})\colon t=(t_k)\in R^m,\ \sum t_k^2=1\right\}. \tag{2}$$

In particular, $\mathrm{lust}(R^{m^2},c_p)\geqslant\frac{1}{8}m^{|1/2-1/p|}$.

Proof. Let us consider elements of E as $(m\times m)$-matrices. We apply the Theorem to the space E (and $n=m^2$). Choose $(x_k)_{k=1}^{m^2}$ to be a natural basis consisting of matrices with only one nonzero element and G to be the set of all matrices of signs $(\varepsilon_{ij})_{i,j=1}^m$ which may be represented as $\varepsilon_{ij}=\eta_i\delta_j$ for some $(\eta_i),(\delta_j)\in\{-1,1\}^m$. Then (i) is obvious for $K=1$. To prove (ii) for $M=2$ we apply twice the Khinchine inequality (see, e.g., [4]). Hence, by the Theorem, $\mathrm{ubc}(x_k)\leqslant 4\,\mathrm{lust}(E)$.

Now to complete the proof of the Corollary it remains to estimate from below the unconditional constant of the natural basis. To this end, assume first that $m=2^d$ for some positive integer d. Let $w=(w_{ij})_{i,j=1}^m$ be the Walsh matrix of the d-th order (which is an orthogonal matrix consisting only of ±1's). Choose any $(t_j)\in R^m$, $\sum t_j^2=1$, and consider two

$(m\times m)$-matrices z and z' defined by $z_{ij}=t_j$ and $z'_{ij}=t_jw_{ij}$ for $i,j=1,2,\ldots,m$. Then $T_wz=z'$, $T_wz'=z$, $a(z)=n^{1/2}$, $a(z')=n^{1/2}\|(t_j)\|$ and, as a consequence,

$$\|T_w\|\geqslant\max\left(\|(t_j)\|,\ \|(t_j)\|^{-1}\right).$$

Taking the supremum over (t_j) we prove (2) for $m=2^d$ with $\frac{1}{4}$ instead of $\frac{1}{8}$. The case of arbitrary m holds now in a standard way.

Proof of the Theorem. We begin with the following simple observation. Let Y be a B-space and $(y_k)_{k=1}^N$ its basis. Then (1)

$$\begin{aligned}(3)\qquad \operatorname{ubc}(y_k)&=\inf\Big\{C:\ \forall(t_k),(a_k)\in R^N\ \sum|a_kt_k|\leqslant C\Big\|\sum t_ky_k\Big\|\Big\|\sum a_ky_k^*\Big\|\Big\}\\&=\inf\Big\{C:\ \forall y\in Y,\ y^*\in Y^*\ \sum|y_k^*(y)|\,|y^*(y_k)|\leqslant C\|y\|\,\|y^*\|\Big\}.\end{aligned}$$

The first inequality is a direct consequence of the definition of $\operatorname{ubc}(y_k)$, the second one is obtained by reformulating the condition in the brackets.

Hence to prove the Theorem it is enough to show that, for any B-space F, any unconditional basis (e_j) of F, any operators $u\colon E\to F$, $v\colon F\to E$ satisfying $v\circ u=\mathrm{id}_E$, and any $(a_k),(t_k)\in R^n$, we have

$$\sum|a_kt_k|\leqslant K^2M^2\|u\|\,\|v\|\operatorname{ubc}(e_j)\cdot\|x\|\,\|x^*\|,$$

where $x=\sum t_kx_k$ and $x^*=\sum a_kx_k^*$.

We have

$$\begin{aligned}\sum_k|a_kt_k|&=\sum_k|a_kt_k|x_k^*(x_k)=\sum_k|a_kt_k|\Big(\sum_j x_k^*(ve_j)e_j^*(ux_k)\Big)\\&\leqslant\sum_j\sum_k|a_kx_k^*(ve_j)|\,|t_ke_j^*(ux_k)|,\end{aligned}$$

where the second equality follows from the fact that $vux_k=x_k$ for $k=1,2,\ldots,n$. Now, by the Schwartz inequality,

$$\begin{aligned}\sum_k|a_kt_k|&\leqslant\sum_j\Big(\sum_k|a_kx_k^*(ve_j)|^2\Big)^{1/2}\Big(\sum_k|t_ke_j^*(ux_k)|^2\Big)^{1/2}\\&\leqslant M^2\sum_j\operatorname*{Av}_{(\varepsilon_k)\in G}\Big|\sum_k\varepsilon_ka_kx_k^*(ve_j)\Big|\operatorname*{Av}_{(\eta_k)\in G}\Big|\sum_k\eta_kt_ke_j^*(ux_k)\Big|\qquad\text{(by (ii))}\\&=M^2\operatorname*{Av}_{(\varepsilon_k),(\eta_k)\in G}\sum_j\big|v^*(T_\varepsilon^*(x^*))(e_j)\big|\,\big|e_j^*(u(T_\eta(x)))\big|\\&\leqslant M^2\operatorname{ubc}(e_j)\operatorname*{Av}_{(\varepsilon_k),(\eta_k)\in G}\big\|v^*(T_\varepsilon^*(x^*))\big\|\,\big\|u(T_\eta(x))\big\|\qquad\text{(by (3))}\end{aligned}$$

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(1) We denote by (x_n^*), (y_n^*), (e_n^*) the dual bases to the bases (x_n), (y_n), (e_n), respectively.

$$\leqslant M^2 \mathrm{ubc}(e_j)\,\|v^*\|\,\|u\|\,\big[\operatorname*{Av}_{(\varepsilon_k),(\eta_k)\in G} \|T_\varepsilon^*\|\,\|T_\eta\|\big]\,\|x\|\,\|x^*\|$$
$$\leqslant M^2 \mathrm{ubc}(e_j)\cdot\|v\|\,\|u\|\,K^2\,\|x\|\,\|x^*\| \qquad \text{(by (i))}.$$

This completes the proof.

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Reçu par la Rédaction le 20. 12. 1978;
en version modifée le 5. 5. 1979