COLLOQUIUM MATHEMATICUM

VOL. XXIX 1974 FASC. 2

ON PROJECTIVE RESOLUTIONS OF FLAT MODULES

BY

D. SIMSON (TORUŃ)

One of the aims of this paper is the following theorem proved in Section 1:

If M is a flat right R-module and if

$$0 \to K \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$$

is any exact sequence of right R-modules, where $P_0, P_1, \ldots, P_n$ are projective, then K is an $\aleph_n$-directed union of $\aleph_n$-generated projective submodules.

Lazard showed this for $n = 0$ (see [10], Theorem 3.3).

In Section 2 we shall consider the following problem: "When $\aleph$-generated flat modules admit $\aleph$-generated projective resolutions?". We prove that if a ring R has a two-sided nilpotent ideal J such that the ring R/J can be embedded in a right Noetherian ring or in a right perfect ring, then any $\aleph$-generated flat right R-module admits an $\aleph$-generated projective resolution.

In Section 3 we apply results of Section 1 and results of Osofsky [11] to prove that $\aleph_n\text{-}P_R \neq \aleph_{n+1}\text{-}P_R$, $n = -1, 0, 1, \ldots$, for some valuation rings R, where $\aleph_n\text{-}P_R$ is the category of all $\aleph_n$-projective right R-modules (cf. [14]).

Finally, in Section 4 it is shown that if R is a right $\aleph_{a+n}$-Noetherian ring which has all $\aleph_a$-generated right ideals projective, then the right global dimension of R is not greater than $n+1$.

All rings considered in this paper have identity element, subrings have the same identity, and all modules are right unitary. $\mathrm{r.pd}_R M$ denotes the projective dimension of the right R-module M, $|X|$ denotes the cardinality of the set X, and $\aleph_{-1}$ is the cardinality of finite sets, e.g., $|X| \leqslant \aleph_{-1}$ means that X is a finite set.

Throughout this paper $\aleph$ denotes an infinite cardinal number or $\aleph = \aleph_{-1}$.

1. Main results. We recall that a right R-module M is *$\aleph$-presented* if $\operatorname{Ker}(F\to M)$ is $\aleph$-generated for any epimorphism $F\to M$ with $\aleph$-generated free module F.

A basic tool for our considerations is the following

THEOREM 1.1 (Jensen [7], Osofsky [12]). *If M is an $\aleph_n$-presented flat R-module, then*

$$\mathrm{r.pd}_R M \leqslant n+1, \quad n \geqslant -1.$$

The theorem immediately follows from Theorem 1.3 of [12]. Jensen proved this for $n = 0$ (cf. [7], Lemma 2).

We recall that a right R-module M *is $\aleph$-projective* (*$\aleph$-flat*) if, for any $\aleph$-generated ($\aleph$-presented) right R-module X and for any epimorphism $B\to M$, the induced homomorphism

$$\operatorname{Hom}_R(X, B) \to \operatorname{Hom}_R(X, M)$$

is surjective. For characterizations of $\aleph$-projective and $\aleph$-flat modules see [14].

To prove the next proposition we shall need the following

LEMMA 1.2. *If M is an $\aleph$-generated $\aleph_{-1}$-projective module, then M is $\aleph$-presented.*

Proof. Let

$$0 \to K \to F \to M \to 0$$

be an exact sequence, where $K = \operatorname{Ker}(F\to M)$ and F is free with a base I of cardinality not greater than $\aleph$. Let F_η be the free submodule of F generated by a finite subset η of I. Since M is $\aleph_{-1}$-projective, it follows from Theorem 7 of [14] that there exists a homomorphism $u_\eta\colon F_\eta\to K$ such that $\operatorname{Im} u_\eta \supset K\cap F_\eta$. Hence

$$K = \bigcup_\eta \operatorname{Im} u_\eta,$$

where η runs over all finite subsets of I. Then K is $\aleph$-generated and the proof of the lemma is complete.

COROLLARY 1.3. *Let M be an $\aleph_n$-generated right R-module, $n \geqslant -1$. If M is $\aleph_{-1}$-projective, then* $\mathrm{r.pd}_R M \leqslant n+1$.

The corollary immediately follows from the previous lemma and Theorem 1.1.

Let A be a submodule of a right R-module B. We say that A *is a pure submodule of* B if the induced homomorphism

$$A\otimes_R X \to B\otimes_R X$$

is injective for any left R-module X (cf. [6]). It is clear that if B is projective, A is a pure submodule of B if and only if B/A is flat.

If I is a set, $F(I)$ will denote the free right R-module with the free base I.

We are now able to prove

PROPOSITION 1.4. *Let $\aleph$ be an infinite cardinal number and let*

$$\ldots \to F(I_n) \xrightarrow{d_n} F(I_{n-1}) \to \ldots \to F(I_0) \xrightarrow{d_0} M \to 0$$

be an exact sequence with M flat. Moreover, let $K_n = \operatorname{Ker} d_n$. If N and L are $\aleph$-generated submodules of K_n and K_0, respectively, and if $J_1 \subset I_1, \ldots$ $\ldots, J_n \subset I_n$ are subsets of cardinality not greater than $\aleph$, then there exists an exact sequence

$$0 \to N^* \to F(J_n^*) \xrightarrow{d_n^*} F(J_{n-1}^*) \to \ldots \to F(J_1^*) \xrightarrow{d_1^*} L^* \to 0$$

such that

(a) $J_i \subset J_i^* \subset I_i$, $d_i^* = d_i | F(J_i^*)$ *and* $|J_i^*| \leqslant \aleph$ *for* $i = 1, 2, \ldots, n$;

(b) $N \subset N^* \subset K_n$, $L \subset L^* \subset K_0$, *and* N^*, L^* *are $\aleph$-generated*;

(c) L^* *is a pure submodule of* $F(I_0)$.

Proof. We use induction on n.

$n = 0$. Let N and L be $\aleph$-generated submodules of K_0. It is sufficient to show that there exists an $\aleph$-generated submodule L^* of K_0 such that $N \subset L^*$, $L \subset L^*$, and L^* is a pure submodule of $F(I_0)$. Let L_0 be the submodule of K_0 generated by $N \cup L$ and let y_t, $t \in T$, $|T| \leqslant \aleph$, be generators of L_0. It follows from Theorem 4 of [14] for $\aleph = \aleph_{-1}$ that, for any finite subset η of T, there exists a homomorphism $u_\eta \colon F(I_0) \to K_0$ such that $\operatorname{Im} u_\eta$ is finitely generated and $u_\eta(y_t) = y_t$ for all $t \in \eta$. Let L_1 be the submodule of K_0 generated by $\bigcup_\eta \operatorname{Im} u_\eta$, where η runs over all finite subsets of T. Clearly, L_1 is $\aleph$-generated and, for any finitely generated submodule H of L_0, there exists a homomorphism $u \colon F(I_0) \to L_1$ such that $u(h) = h$ for $h \in H$. Continuing in this way, we get an ascending chain of $\aleph$-generated submodules of K_0

$$L_0 \subset L_1 \subset \ldots \subset L_n \subset \ldots \subset F(I_0)$$

such that, for any finitely generated submodule H of L_n, there exists a homomorphism $u \colon F(I_0) \to L_{n+1}$ which is the identity on H. Then, it follows from Theorem 4 of [14] that the $\aleph$-generated module

$$L^* = \bigcup_{i=0}^{\infty} L_i$$

is a pure submodule of $F(I_0)$ ([1]).

([1]) This technique is similar to that in [12].

Now assume $n > 0$ and the proposition holds for $n-1$. Select a set $J_n^0 \subset I_n$ with $|J_n^0| \leqslant \aleph$ such that $J_n \subset J_n^0$ and $N \subset F(J_n^0)$. Then, by the induction hypothesis, we get an exact sequence

$$0 \to H^0 \to F(J_{n-1}^0) \to \ldots \to F(J_1^0) \to L^0 \to 0$$

with $H^0 = \big(d_n F(J_n^0)\big)^*$, $J_{n-1}^0 = J_{n-1}^*, \ldots, J_1^0 = J_1^*$, $L^0 = L^*$, such that conditions (a)-(c) are satisfied for $n-1$.

Select a set $J_n^1 \subset I_n$ with $|J_n^1| \leqslant \aleph$ such that $J_n^0 \subset J_n^1$ and $H^0 \subset d_n F(J_n^1)$. Continuing in this way, we get an ascending chain of sets

$$J_j^0 \subset J_j^1 \subset \ldots \subset J_j^s \subset \ldots \subset I_j, \quad j = 1, 2, \ldots, n, \ |J_j^s| \leqslant \aleph,$$

and an ascending chain of exact sequences

$$\begin{array}{ccccccccccc}
0 & \to & H^0 & \to & F(J_{n-1}^0) & \to \ldots \to & F(J_1^0) & \to & L^0 & \to & 0 \\
 & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \\
0 & \to & H^s & \to & F(J_{n-1}^s) & \to \ldots \to & F(J_1^s) & \to & L^s & \to & 0 \\
 & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

satisfying conditions (a)-(c) and such that

$$d_n F(J_n^s) \subset H^s \subset d_n F(J_n^{s+1}) \quad \text{for all } s.$$

Set

$$H^* = \bigcup_{i=0}^{\infty} H^i, \quad L^* = \bigcup_{i=0}^{\infty} L^i, \quad J_j^* = \bigcup_{i=0}^{\infty} J_j^i \ \text{ for } j = 1, 2, \ldots, n,$$

and let

$$N^* = \operatorname{Ker}\big(F(J_n^*) \xrightarrow{d_n^*} H^*\big).$$

It is clear that

$$F(J_n^*) \xrightarrow{d_n^*} H^*$$

is an epimorphism. Futhermore, the sequence

$$0 \to N^* \to F(J_n^*) \xrightarrow{d_n^*} F(J_{n-1}^*) \to \ldots \to F(J_1^*) \xrightarrow{d_1^*} L^* \to 0$$

is exact and L^* is a pure submodule of $F(I_0)$ as a union of an ascending chain of pure submodules L^i. Since, clearly, $N \subset N^*$, $L \subset L^*$, and L^* is $\aleph$-generated, it remains to show that N^* is $\aleph$-generated.

Since $F(I_0)/L^*$ is flat, it follows from Corollary 8 of [14] for $\aleph = \aleph_{-1}$ that

$$L^* \text{ and } K_j^* = \operatorname{Ker}\big(F(J_j^*) \xrightarrow{d_j^*} F(J_{j-1}^*)\big), \quad j = 1, 2, \ldots, n,$$

are $\aleph_{-1}$-projective. But L^* and $F(J_j^*)$ are $\aleph$-generated, and so, by Lemma 1.2, $K_1^*, K_2^*, \ldots, K_n^* = N^*$ are also $\aleph$-generated. Thus the proof is completed.

THEOREM 1.5. *Let*

$$(*) \qquad 0 \to K \to P_n \xrightarrow{d_n} P_{n-1} \to \ldots \to P_0 \xrightarrow{d_0} M \to 0$$

be an exact sequence with M flat and $P_0, P_1, \ldots, P_n$ projective. Then K is an $\aleph$-directed union of $\aleph_n$-generated projective submodules. In particular, K is $\aleph_n$-projective.

Proof. Adding to $(*)$ sequences of the form

$$0 \to P' \xrightarrow{1} P' \to 0$$

with P' projective, we derive an exact sequence

$$0 \to K \to F(I_n) \to F(I_{n-1}) \to \ldots \to F(I_0) \to M' \to 0$$

with M' flat.

Let N be an $\aleph_n$-generated submodule of K. Then, it follows from Proposition 1.4, for $\aleph = \aleph_n$, that there exists an exact sequence

$$0 \to N^* \to F(J_n) \to F(J_{n-1}) \to \ldots \to F(J_1) \to H \to 0$$

such that $|J_i| \leqslant \aleph_n$ for $i = 1, 2, \ldots, n$, N^* and H are $\aleph_n$-generated, and H is a pure submodule of $F(I_0)$. Hence H is a pure submodule of $F(J_0)$ for some $J_0 \subset I_0$ with $|J_0| \leqslant \aleph_n$. Then $F(J_0)/H$ is flat and $\aleph_n$-presented, and thus, by Theorem 1.1,

$$\mathrm{r.pd}_R F(J_0)/H \leqslant n+1.$$

Hence N^* is projective and the proof of the theorem is complete.

2. When $\aleph$-generated flat modules admit $\aleph$-generated projective resolutions?

PROPOSITION 2.1. *Any $\aleph$-generated $\aleph_{-1}$-projective module admits an $\aleph$-generated projective resolution.*

The proposition follows from Lemma 1.2 and Corollary 8 of [14].

The class of $\aleph_{-1}$-projective modules is rather big. For instance,

(a) directed unions of projective modules are $\aleph_{-1}$-projective; in particular, pure submodules of projective modules and localizations R_S, whenever S is a multiplicatively closed subset of the ring R consisting of non-zero-divisors, are $\aleph_{-1}$-projective;

(b) if R is a subring of a semi-fir ring Q (i.e., Q is a domain with the invariant basis number property for which any finitely generated right ideal is free), then it follows from Theorem 2B of [8] and from Theorem 1.2 of [15], for $\aleph = \aleph_{-1}$, that any flat right R-module is $\aleph_{-1}$-projective;

(c) if R is a subring of a right Noetherian ring or of a right perfect ring, then any flat right R-module is $\aleph_{-1}$-projective (cf. [15]).

THEOREM 2.2. *If M is an $\aleph$-generated flat right R-module and there is satisfied at least one of the conditions*

(i) *R is a subring of a ring S and $M \otimes_R S$ admits an $\aleph$-generated S-projective resolution,*

(ii) *there exists a two-sided nilpotent ideal J in R such that $M \otimes_R R/J$ admits an $\aleph$-generated R/J-projective resolution,*

then M admits an $\aleph$-generated projective resolution.

Proof. We have a short exact sequence

$$0 \to K \xrightarrow{i} F \to M \to 0,$$

where F is free and $\aleph$-generated. By Corollary 8 of [14], K is $\aleph_{-1}$-projective and thus, in virtue of Proposition 2.1, it is sufficient to prove that K is $\aleph$-generated.

Suppose that condition (i) is satisfied. Since $\mathrm{Tor}_1^R(M, S) = 0$ and K is flat, we derive the exact sequences

$$0 \to K \otimes_R S \to F \otimes_R S \to M \otimes_R S \to 0$$

and

$$0 \to K \xrightarrow{j} K \otimes_R S,$$

where j is defined by $j(k) = k \otimes 1$. Since $F \otimes_R S$ is an $\aleph$-generated free S-module, $K \otimes_R S$ is $\aleph$-generated because $M \otimes_R S$ is $\aleph$-presented by (i). Let $\{k_p \otimes 1\}_{p \in I}$, $|I| \leqslant \aleph$, $k_p \in K$, be a set of generators of the S-module $K \otimes_R S$ and let T be the set of all finite subsets of I. It follows from Theorem 4d of [14] that, for any $t \in T$, there exists a homomorphism $u_t \colon F \to K$ such that $u_t i(k_p) = k_p$ for all $p \in t$, and $\mathrm{Im}\, u_t$ is finitely generated. Then, it is sufficient to show that K is generated by $\bigcup_{t \in T} \mathrm{Im}\, u_t$ because $|T| \leqslant \aleph$.

Let $k \in K$. If

$$j(k) = \sum_{p \in t} k_p \otimes s_p, \qquad s_p \in S,\ t \in T,$$

then, in view of the equality $j u_t i = (u_t i \otimes 1) j$, we have

$$j u_t i(k) = \sum_{p \in t} u_t i(k_p) \otimes s_p = j(k),$$

whence $u_t i(k) = k$ because j is a monomorphism. Hence $k \in \mathrm{Im}\, u_t$ and the proof of the theorem is complete in case (i). In case (ii) the theorem is a consequence of the following

LEMMA 2.3. *Let J be a two-sided nilpotent ideal in the ring R and let M be a right R-module. Then*

(a) *if the right R/J-module M/MJ is $\aleph$-generated, then M is $\aleph$-generated*;

(b) *if M is flat and M/MJ is an $\aleph$-presented R/J-module, then M is $\aleph$-presented.*

Proof. If $\{m_t+MJ\}_{t\in T}$, $m_t\in M$, $|T|\leqslant\aleph$, is a set of generators of the right R/J-module M/MJ and if N is the R-submodule of M generated by the elements m_t, $t\in T$, then

$$M = N+MJ = N+MJ^2 = \ldots = N$$

because J is nilpotent. Therefore (a) follows.

If M/MJ is $\aleph$-presented R/J-module, then M is $\aleph$-generated by (a), and thus there exists an exact sequence

$$0\to K\to F\to M\to 0,$$

where F is free and $\aleph$-generated. Since $\mathrm{Tor}_1^R(M, R/J) = 0$, we derive the exact sequence

$$0\to K/KJ\to F/FJ\to M/MJ\to 0,$$

where K/KJ is $\aleph$-generated. By (a), K is $\aleph$-generated and the proof is completed.

COROLLARY 2.4. *Let J be a two-sided nilpotent ideal in the ring R. If the ring R/J can be embedded in a right Noetherian ring or in a right perfect ring or in a semi-fir, then*

(a) *any $\aleph$-generated flat R-module admits an $\aleph$-generated projective resolution*;

(b) $\mathrm{r.pd}_R M\leqslant n+1$ *for any $\aleph_n$-generated flat R-module M.*

Proof. In view of the remark following Proposition 2.1, (a) is a consequence of Proposition 2.1 and of Theorem 2.2, and (b) follows from (a) and from Corollary 1.3.

3. $\aleph_n$-projective modules. Consider the sequence of categories

$$\mathrm{Fl}_R\supset\aleph_{-1}\text{-}P_R\supset\aleph_0\text{-}P_R\supset\ldots\supset\aleph_n\text{-}P_R\supset\ldots\supset P_R,$$

where Fl_R, $\aleph_n\text{-}P_R$ and P_R denote the categories of all flat, $\aleph_n$-projective and projective right R-modules, respectively.

It follows from [3] that $\mathrm{Fl}_R = P_R$ iff R is a right perfect ring. By the remark following Proposition 2.1, $\mathrm{Fl}_R = \aleph_{-1}\text{-}P_R$ whenever the ring R can be embedded in a right Noetherian ring, or in a right perfect ring, or in a semi-fir. In this section we give examples of rings such that $\aleph_n\text{-}P_R \neq\aleph_{n+1}\text{-}P_R$ for $n\geqslant -1$.

We start with the following

PROPOSITION 3.1. *If $\mathrm{Fl}_R = \aleph_{-1}\text{-}P_R$, then R has no infinite set of orthogonal idempotents.*

Proof. Let X be a set of orthogonal idempotents of R. If I is the ideal generated by X, then $R/I\in\mathrm{Fl}_R = \aleph_{-1}\text{-}P_R$ and, therefore, R/I is projective.

It then follows that X is finite and the proof of the proposition is complete.

PROPOSITION 3.2. *If R is a von Neumann regular ring, then*

(i) *any countably generated $\aleph_{-1}$-projective R-module is projective,*

(ii) $\aleph_{-1}P_R = \aleph_0\text{-}P_R$.

Proof. Let M be a countably generated $\aleph_{-1}$-projective right R-module. Then M can be written as an ascending union of finitely generated submodules $M_1 \subset M_2 \subset M_3 \subset \ldots$ By the $\aleph_{-1}$-projectivity of M, M_i can be embedded in a free module and, therefore, it is projective (see [5], Chapter I, Proposition 6.2). Thus the flat modules M_{i+1}/M_i are finitely presented and, hence, projective by [10]. It then follows that

$$M \simeq \bigoplus_{i=0}^{\infty} M_{i+1}/M_i$$

is projective and (i) is proved. (ii) follows from (i) and from the fact that any $\aleph_{-1}$-projective module is an $\aleph_0$-directed union of countably generated $\aleph_{-1}$-projective submodules (see Theorem 9 of [14]).

THEOREM 3.3. *Let R be a valuation ring such that, for every $n = 0, 1, \ldots,$ there exists an ideal I^n which is generated by $\aleph_{n+1}$ but not by a fewer number of elements* (cf. [11]). *Then $\aleph_n\text{-}P_R \neq \aleph_{n+1}\text{-}P_R$ for $n \geqslant -1$.*

Proof. Observe that I^n is $\aleph_{-1}$-projective as a directed union of finitely generated (so free) ideals, but, in view of Theorem A of [12], I^n is not projective. Then, in virtue of Corollary 3a of [14], I^{-1} is not $\aleph_0$-projective, and thus the theorem follows for $n = -1$.

Now let $n \geqslant 0$. By Proposition 2.1, there exists an exact sequence

$$0 \to K \to F_n \to \ldots \to F_0 \to I^n \to 0,$$

where $F_0, \ldots, F_n$ are free and K is $\aleph_{n+1}$-generated.

Since $\mathrm{pd}_R I^n = n+2$ by Theorem A of [11], K is not projective and, therefore, using Corollary 3a of [14], we conclude that K is not $\aleph_{n+1}$-projective. On the other hand, it follows from Theorem 1.5 that K is $\aleph_n$-projective. Thus the proof of the theorem is complete.

Remark. Using results of Balcerzyk [2] and Pierce [13], one can also prove that $\aleph_n\text{-}P_R \neq \aleph_{n+1}\text{-}P_R$ for $n \geqslant 0$ provided R is the free Boolean ring on $\aleph_{\omega_0}$ generators or $R = K(T)$ is the group ring, where T is a direct sum of finite cyclic groups, $|T| = \aleph_{\omega_0}$, and K is a field such that $mK = K$ whenever m is the order of an element in T.

Example. Let R be the complete direct product of $\aleph_{a+1}$ copies of the field Z_2 and let I be the ideal in R generated by elements $e_\eta = \{e^i_\eta\}_{i<\omega_{a+1}}$, $\eta < \omega_{a+1}$, defined by

$$e^i_\eta = \begin{cases} 0 & \text{for } i \geqslant \eta, \\ 1 & \text{for } i < \eta. \end{cases}$$

Since I cannot be generated by less than $\aleph_{\alpha+1}$ elements, using the well-known Kaplansky's argument, one can prove that I is not projective; so I is not $\aleph_{\alpha+1}$-projective by Corollary 3 of [14]. Observe also that I is $\aleph_\alpha$-projective as the $\aleph_\alpha$-directed union of the projective ideals Re_η, $\eta < \omega_{\alpha+1}$. Then we have $\aleph_\alpha\text{-}P_R \neq \aleph_{\alpha+1}\text{-}P_R$. Moreover, $\aleph_\alpha\text{-}\mathrm{Fl}_R \neq \aleph_\alpha\text{-}P_R$. In fact, by Theorem 4 of [14], the module $R/I = \varinjlim R/Re_\eta$ is $\aleph_\alpha$-flat but it is not $\aleph_\alpha$-projective because I cannot be generated by less than $\aleph_{\alpha+1}$ elements.

4. On the global dimension of $\aleph_\alpha$-hereditary rings. A ring R is called *right $\aleph$-hereditary* if every $\aleph$-generated right ideal in R is projective (cf. [4]). Examples of $\aleph_0$-hereditary rings are regular rings in the sense of von Neumann (cf. [9]).

A right R-module M is called *$\aleph$-injective* if, for any $\aleph$-generated right ideal $I \subset R$, the induced homomorphism $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I, M)$ is surjective.

Using the same type of arguments as in the proof of Theorem 5.4, Chapter I in [5], one can prove the following

THEOREM 4.1. *For any ring R, the following conditions are equivalent*:

(i) *R is right $\aleph$-hereditary*;

(ii) *any $\aleph$-generated submodule of a right projective R-module is projective*;

(iii) *any factor-module of a right injective R-module is $\aleph$-injective.*

COROLLARY 4.2. *Let R be a right $\aleph$-hereditary ring. Then*

(i) *a right R-module M is $\aleph$-projective if and only if M is an $\aleph$-directed union of its $\aleph$-generated projective submodules*;

(ii) *a submodule of an $\aleph$-projective module is $\aleph$-projective.*

The proof is immediate from the definition and Theorem 4.1.

We are now able to prove

PROPOSITION 4.3. *Let R be a right $\aleph_\alpha$-hereditary ring and let $n \geqslant 0$ be an integer. If M is a right $\aleph_\alpha$-projective R-module generated by $\aleph_{\alpha+n}$ elements, then $\mathrm{r.pd}_R M \leqslant n$.*

Proof. We use induction on n. It is clear that the proposition holds for $n = 0$. Suppose $n \geqslant 1$ and let m_ξ, $\xi < \omega_{\alpha+n}$, be generators of M. If M_η, $\eta < \omega_{\alpha+n}$, is a submodule of M generated by all elements m_ξ, $\xi < \eta$, then, by Corollary 4.2, $\big(\bigcup_{\eta<\tau} M_\eta\big)$ is $\aleph_\alpha$-projective for any $\tau < \omega_{\alpha+n}$ and, therefore,

$$\mathrm{r.pd}_R \Big(\bigcup_{\eta<\tau} M_\eta\Big) \leqslant n-1$$

by the induction hypothesis. Hence

$$\mathrm{r.pd}_R \Big(M_\tau / \bigcup_{\eta<\tau} M_\eta\Big) \leqslant n \quad \text{for all } \tau < \omega_{\alpha+n},$$

and it follows from Proposition 3 of [1] that $\mathrm{r.pd}_R M \leqslant n$. Thus the proposition follows.

COROLLARY 4.4. *If R is a right $\aleph_a$-hereditary and right $\aleph_{a+n}$-Noetherian ring, then* $\mathrm{r.gl.dim} R \leqslant n+1$.

Proof. By Corollary 4.2, every right ideal I of R is $\aleph_a$-projective. It then follows from Proposition 4.3 that $\mathrm{r.pd}_R I \leqslant n$. Hence $\mathrm{r.pd}_R(R/I) \leqslant n+1$, and thus the corollary follows from [1].

For $a = -1$ the corollary follows from [12].

REFERENCES

[1] M. Auslander, *On the dimension of modules and algebras III*, Nagoya Mathematical Journal 9 (1955), p. 67-77.

[2] S. Balcerzyk, *The global dimension of the group ring of abelian groups III*, Fundamenta Mathematicae 67 (1970), p. 241-250.

[3] H. Bass, *Finitistic dimension and homological generalization of semi-primary rings*, Transactions of the American Mathematical Society 95 (1960), p. 466-488.

[4] G. M. Bergman, *Hereditary commutative rings and centers of hereditary rings*, Proceedings of the London Mathematical Society 23 (1971), p. 214-236.

[5] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton 1956.

[6] P. M. Cohn, *On the free product of associative rings*, Mathematische Zeitschrift 71 (1958), p. 380-398.

[7] C. U. Jensen, *On homological dimensions of rings with countably generated ideals*, Mathematica Scandinavica 18 (1966), p. 97-105.

[8] — *Some cardinality questions for flat modules and coherence*, Journal of Algebra 12 (1969), p. 231-241.

[9] I. Kaplansky, *On the dimension of modules and algebras X*, Nagoya Mathematical Journal 13 (1958), p. 85-88.

[10] D. Lazard, *Autour de la platitude*, Bulletin de la Société Mathématique de France 97 (1969), p. 81-128.

[11] B. L. Osofsky, *Global dimension of valuation rings*, Transactions of the American Mathematical Society 126 (1967), p. 136-149.

[12] — *Upper bounds on homological dimensions*, Nagoya Mathematical Journal 32 (1968), p. 315-322.

[13] R. S. Pierce, *The global dimension of Boolean rings*, Journal of Algebra 7 (1967), p. 91-99.

[14] D. Simson, *$\aleph$-flat and $\aleph$-projective modules*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 20 (1972), p. 109-114.

[15] — *On the structure of flat modules*, ibidem 20 (1972), p. 115-120.

INSTITUTE OF MATHEMATICS
N. COPERNICUS UNIVERSITY, TORUŃ

Reçu par la Rédaction le 8. 6. 1972