COLLOQUIUM MATHEMATICUM

VOL. LV 1988 FASC. 2

ON THE LATTICE BOUNDARY OF MARKOV OPERATORS

BY

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Let X be a compact Hausdorff space. By $C(X)$ we denote the space of all real-valued continuous functions on X. A linear operator $T: C(X) \to C(X)$ is called *Markov* if $T1 = 1$ and $f \geqslant 0 \Rightarrow Tf \geqslant 0$. The main purpose of this note is to investigate lattice properties of the space C_T of T-invariant (continuous) functions. Davies in [2] and Rębowski in [3] have studied lattice properties of C_T for strongly ergodic Markov operators. We extend [2] and [3] and obtain similar results under the weaker assumption that the invariant functions form a lattice for the ordering inherited from $C(X)$. In some of our results even this lattice condition is omitted. In the first part of our note we consider arbitrary Markov operators. We introduce here a T-invariant subset ∂T of the conservative set W of T (see [4] for the definition). A simple consequence of our Theorem 1 is that two different T-invariant functions are different on ∂T (Corollary 2). If strong ergodicity of T is assumed, then the boundary ∂T coincides with the one investigated in [2]. If C_T forms a lattice, ∂T also coincides with the boundary considered in [3]. In the second part of the paper we assume that C_T is a lattice. Under this assumption we characterize those functions on ∂T which have unique T-invariant continuous extension (Corollary 3). Our last results describe those Markov operators T for which there exists a Markov projection P such that $C_P = C_T$ (Corollaries 5 and 6).

The notation and terminology follow [1] and [4]. We identify the dual space of $C(X)$ with the space $M(X)$ of all Radon measures on X. The convex and w*-compact set of all probability Radon measures on X is denoted by $P(X)$. A measure $\mu \in P(X)$ is called *T-invariant* if $T^* \mu = \mu$. The set $P_T(X)$ of all invariant probabilities is a nonempty convex and w*-compact subset of $P(X)$. A nonempty closed subset Y of X is said to be *invariant* if $y \in Y$ implies $T^* \delta_y(Y) = 1$. Recall that T is called *strongly ergodic* (*uniformly mean stable* in [4]) if the Cesàro means

$$A_n f = n^{-1}(I + T + \ldots + T^{n-1})f$$

converge uniformly for each f in $C(X)$. The structure of C_T, for T strongly ergodic, has been studied by several authors (see [2]–[5]).

Recall that a compact convex subset K of a locally convex vector space is called a *Bauer simplex* if $\operatorname{ex} K$ is closed and every point in K is the barycenter of a unique Radon probability measure carried by $\operatorname{ex} K$. Let E be a closed linear subspace of $C(X)$ containing the constants. We let

$$E^{\perp} = \{\mu \in M(X) \colon \forall f \in E \ \int f d\mu = 0\}.$$

Denote by $\varkappa$ the canonical projection of $M(X)$ onto the quotient Banach space $M(X)/E^{\perp}$. Every $f \in E$ defines a linear functional $\langle f, \cdot \rangle$ on the quotient space by the formula

$$\langle f, \varkappa(\mu)\rangle = \int f d\mu.$$

From the general theory of Banach spaces it is known that under this duality $M(X)/E^{\perp}$ is the Banach dual of E. The mapping $\varkappa$ is clearly weak*-weak* continuous. Now set $Q = \varkappa(P(X))$ and for every $f \in E$ denote by $\hat{f}$ the function $\hat{f}(q) = \langle f, q\rangle$, $q \in Q$. Now $\hat{f}$ is clearly weak* continuous and affine on Q, i.e., $\hat{f} \in A(Q)$. The following lemma is essentially known but for the sake of completeness we present a proof here.

LEMMA 1. *The mapping $f \to \hat{f}$ is a linear order-preserving isometry of E onto the space $A(Q)$ of all weak* continuous affine functions on Q.*

Proof. Since

$$\|f\| = \sup\{|\int f d\mu| \colon \mu \in P(X)\} \quad \text{for every } f \in C(X),$$

$f \to \hat{f}$ is clearly a linear order-preserving isometry of E into $A(Q)$. We prove that every $F \in A(Q)$ is of the form $\hat{f}$ for some $f \in E$. Given F, define $f(x) = F(\varkappa(\delta_x))$. Since both $\varkappa$ and F are weak* continuous, $f \in C(X)$. Every $\mu \in P(X)$ can be viewed as a probability Radon measure on the extreme points of $P(X)$. Now

$$F(\varkappa(\mu)) = F \circ \varkappa\left(\int \delta_x d\mu(x)\right) = \int F \circ \varkappa(\delta_x) d\mu(x) = \int f d\mu,$$

which implies $\int f d\mu = \int f d\nu$ whenever $\varkappa(\mu) = \varkappa(\nu)$ and both μ and ν are probability measures on X. Suppose $f \notin E$. By the Hahn–Banach theorem, there exists an $\eta \in E^{\perp}$ with $\int f d\eta \neq 0$. Since E contains the constants, $\eta = \eta_1 - \eta_2$, $\eta_i \geqslant 0$, and $\eta_1(X) = \eta_2(X)$. Now we may clearly assume that $\eta_i \in P(X)$. We have

$$\varkappa(\eta_1) = \varkappa(\eta_2) \quad \text{and} \quad \int f d\eta_1 \neq \int f d\eta_2,$$

a contradiction. The equality $\hat{f} = F$ is clear.

We recall that $A(Q)$ is a lattice if and only if Q is a Bauer simplex (see [1], p. 103). Letting $E = C_T$ we obtain the following result:

COROLLARY 1. *C_T is a lattice if and only if Q is a Bauer simplex.*

Our Lemma 1 and Corollary 1 should be compared with Theorem 3 in

[2] where, for strongly ergodic T, the role of Q is played by $K = P_T(X)$ and a similar isomorphism between C_T and $A(K)$ is obtained (see also the remarks following Corollary 2 below). For the rest of this paper we shall assume that $E = C_T$.

THEOREM 1. *Let T be a Markov operator on $C(X)$ and*

$$\partial T = \mathrm{cl}\{x \in X: \varkappa(\delta_x) \in \mathrm{ex}\, Q\}.$$

Then ∂T is T-invariant and

$$\sup_{x\in X} f(x) = \sup_{x\in\partial T} f(x) \quad \textit{for every } f \in C_T.$$

Proof. We show that $\operatorname{supp} T^* \delta_x \subseteq \partial T$ for $\varkappa(\delta_x) = q \in \mathrm{ex}\, Q$. Since $\varkappa^{-1}(q)$ is a convex compact and extremal subset of $P(X)$, there exists a closed subset $X_x \subseteq X$ such that $\varkappa^{-1}(q) = P(X_x)$. We have $\varkappa(T^* \delta_x) = q$, so

$$\operatorname{supp} T^* \delta_x \subseteq X_x \subseteq \partial T.$$

Now by the weak* continuity of the map $x \to T^* \delta_x$ it follows that ∂T is T-invariant. As before, for every $q \in \mathrm{ex}\, Q$, $\varkappa^{-1}(q) = P(X_q)$ for some closed subset X_q of X. Given $f \in C_T$ the affine function $\hat{f}$ attains its supremum on $\mathrm{ex}\, Q$. Using these two observations we obtain

$$\begin{aligned}\sup\{f(x): x \in \partial T\} &= \sup\{f(x): \varkappa(\delta_x) \in \mathrm{ex}\, Q\}\\ &= \sup\{\textstyle\int f(x)\,\mu(dx): \varkappa(\mu) \in \mathrm{ex}\, Q\}\\ &= \sup\{\hat{f}(q): q \in \mathrm{ex}\, Q\}\\ &= \sup\{\hat{f}(q): q \in Q\} = \sup\{f(x): x \in X\}.\end{aligned}$$

COROLLARY 2. *If $f, g \in C_T$ and $f|_{\partial T} = g|_{\partial T}$, then $f = g$.*

Let us observe that if T is strongly ergodic, then

$$\varkappa(\mu) = \varkappa(\lim A_n^* \mu) \quad \text{for every } \mu \in P(X),$$

so $\mu \to \varkappa(\mu)$ is an affine isomorphism between $P_T(X)$ and Q. Moreover, using [4], it can easily by shown that the following conditions are equivalent:

(a) T is strongly ergodic,

(b) $\varkappa$ restricted to $P_T(X)$ establishes an affine isomorphism of $P_T(X)$ and Q.

Consequently, if T is strongly ergodic, then ∂T is the union of all invariant cells of $\mathscr{P}$ (equivalently, $\partial T = W$), where (as in [4]) $\mathscr{P}$ denotes the partition of X into the level sets of C_T. Thus, using [2] (Theorem 9), for strongly ergodic T we have

$$\partial T = \{x \in X: |f(x)| = \lim_{n\to\infty} A_n|f|(x) \text{ for } f \in C_T\}.$$

Even without strong ergodicity we have the following extension of Theorem 9 in [2] providing, if C_T is a lattice, an intrinsic description of ∂T (in [3] this is the definition of ∂T).

THEOREM 2. *If C_T forms a lattice in $C(X)$, then*

$$\partial T = \{x \in X: \ \forall f \in C_T \ \operatorname{mod}(f)(x) = |f(x)|\},$$

where $\operatorname{mod}(f)$ *is the lattice modulus in* C_T.

Proof. Let x be such that $\varkappa(\delta_x) \in \operatorname{ex} Q$ and let $f \in C_T$. Using Lemma 1 we have

$$(\operatorname{mod}(f))\hat{} = \operatorname{mod}(\hat{f}).$$

Now, Q is a Bauer simplex, so by the Bauer theorem (see [1], Theorem II.4.3) we have

$$|f(x)| = |\hat{f}(\varkappa(\delta_x))| = \operatorname{mod}(\hat{f})(\varkappa(\delta_x)) = \operatorname{mod}(f)(x).$$

To prove the converse inclusion we consider a point x such that $\operatorname{mod}(f)(x) = |f(x)|$ for every $f \in C_T$. Using Lemma 1 again, for every $F \in A(Q)$ we have

$$\operatorname{mod}(F)(\varkappa(\delta_x)) = |F|(\varkappa(\delta_x)).$$

By the theory of Bauer simplexes, $\varkappa(\delta_x) = \operatorname{ex} Q$ and the proof is completed.

Remark 1. If C_T is a lattice, then by the Bauer theorem $\operatorname{ex} Q$ is closed, and in the definition of ∂T the closure can be dropped.

By $\mathscr{D} \cap \partial T$ we mean the restriction of the partition $\mathscr{D}$ to the set ∂T. For every $x \in \partial T$ by $D(x)$ we denote the cell D of $\mathscr{D} \cap \partial T$ with $x \in D$. Now we characterize the lattice condition in terms of extensions of functions from ∂T.

COROLLARY 3. *For any Markov operator T the following are equivalent*:

(i) *C_T is a lattice*;

(ii) *every $f \in C(\partial T)$ which is constant on the cells of $\mathscr{D} \cap \partial T$ extends uniquely to some $\bar{f} \in C_T$.*

Proof. (i) $\Rightarrow$ (ii). For every $x \in \partial T$ define $F \in C(\operatorname{ex} Q)$ by $F(\varkappa(\delta_x)) = f(x)$. Since Q is a Bauer simplex, F extends uniquely to some affine continuous function $\bar{F}$ on Q (see [1], p. 105). Consequently, there exists a function $\bar{f} \in C_T$ such that $\hat{\bar{f}} = \bar{F}$. The equality $\bar{f}|_{\partial T} = f$ is clear and, by Lemma 1, we have the uniqueness.

(ii) $\Rightarrow$ (i). By the Bauer theorem it is enough to show that every continuous function F on $\operatorname{ex} Q$ can be uniquely extended to an affine continuous function on Q. We put

$$\bar{F}(\varkappa(\mu)) = \int \bar{f} \, d\mu,$$

where $f = F(\varkappa(\delta_x))$ for $x \in \partial T$.

COROLLARY 4. *If C_T is a lattice, then there exists a natural order-preserving isometry between $C_T|_{\partial T}$ and $C(\operatorname{ex} Q)$.*

The next results are similar to Sine's results in [4] (Theorems 4.2 and 4.3). Using our Corollary 5 we can easily give an example of a Markov operator T such that C_T forms a lattice but there is no Markov projection P

with $C_P = C_T$. Moreover, by Corollary 6, no such C_T can separate the points of X.

COROLLARY 5. *Let T be a Markov operator on $C(X)$. Then the following are equivalent:*

(i) *there exists a Markov projection P acting on $C(X)$ such that $C_T = C_P$;*

(ii) *C_T is a lattice and there exists a family of probability measures $\{m_D\}_{D\in\mathcal{D}\cap\partial T}$ such that the mapping $x \to m_{D(x)}$ from ∂T to $P(X)$ is weak* continuous and satisfies $m_{D(x)}(D(x)) = 1$.*

Proof. (i) $\Rightarrow$ (ii). Since C_P is a lattice, we only have to show the second condition. Put $m_{D(x)} = P^* \delta_x$, where $x \in \partial T$. Since $P^* \delta_y = P^* \delta_x$ whenever x, y are in the same cell of $\mathcal{D}$ (see [4]), $m_{D(x)}$ is well defined and weak* continuous. Since $\partial T = \partial P$ by Theorem 2 and $D(x)$ is P-invariant, we have

$$m_{D(x)}(D(x)) = 1.$$

(ii) $\Rightarrow$ (i). For $f \in C(X)$ we define $f' \in C(\partial T)$ by

$$f'(x) = \int f(y)\, m_{D(x)}(dy)$$

and put $Pf = \overline{f'}$, where $\overline{f'}$ is the extension of f' as in Corollary 3. Now it is easy to verify that P is a Markov projection and $C_T = C_P$.

As a simple version of the above we have

COROLLARY 6. *If C_T separates the points of X, then there exists a Markov projection P with $C_T = C_P$ if and only if C_T is a lattice.*

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Reçu par la Rédaction le 12. 4. 1985