COLLOQUIUM MATHEMATICUM

VOL. XLVII 1982 FASC. 1

THE INDECOMPOSABILITY OF $H^*(G/B, L)$

BY

WALTER LAWRENCE GRIFFITH, JR. (ST. LOUIS, MISSOURI)

Since Bott proved the fundamental theorem concerning the representation theory of the semi-simple linear algebraic groups G on the cohomology spaces of line bundles on the associated flag variety G/B over the complex numbers, a search for a similar result over an algebraically closed field of positive characteristic has been made. The main result of this paper* is the indecomposability of $H^q(G/B, L)$ for appropriate L and q.

Bott's result [3] essentially is in two parts: a vanishing theorem which states that, for a given L, $H^q(L) \neq (0)$ for at most one q, and a result showing that the representation of G on this $H^q(L)$ is equivalent to one on $H^0(L')$ for an explicitly specified L', hence it is irreducible. The first part is false in characteristic p ([6], [7]), although for line bundles in the dominant Weyl chamber it remains true ([1], [8], [9]). The equivalence of representations and the irreducibility fail badly in characteristic p; the purpose of this paper is to show indecomposability remains.

Unless otherwise specified, throughout this paper K will denote an algebraically closed field of characteristic $p > 0$. For other notation see the following definition:

Definition 1. B will denote a Borel subgroup of G, P a parabolic subgroup, L a line bundle on G/B, and $H^q(L)$ its q-th cohomology vector space in the sense of Grothendieck.

Choose some P containing B. There is a locally trivial fibration $t_P: G/B \to G/P$ with fibers isomorphic to P/B.

Fix L. L is characterized up to isomorphism by its degrees (see [1]) $i_1, \ldots, i_{n-1}$. It may be assumed that none of the degrees is -1, else $H^q(L) = (0)$ (see [9]). Consider the set of roots corresponding to the negative degrees. This set determines a parabolic subgroup containing B. Call it P_L and call the corresponding fibration t_L. Let $s = \dim P_L/B$.

---

* Partially supported by NSF Grant #MCS 76-09157.

THEOREM 1. *The natural action of G on $H^s(G/B, L)$ is indecomposable if $R^s t_{L^*}(L)$ is generated by its global sections.*

COROLLARY. *$H^q(G/B, L)$ is indecomposable for any L if* $q = 0, 1$, $\dim G/B - 1$, *or* $\dim G/B$.

First we prove two lemmas.

LEMMA 1. (i) $H^q(L) = (0)$ *if* $q < s$.

(ii) *There exists a P_f-equivariant map γ_f: $H^s(L) \to H^s(L_f)$, where f is a fiber of t_L stabilized by P_f (a conjugate of P_L).*

Proof. (i) Note that P_L is itself a semi-simple linear algebraic group. Hence $H^q(P_L/B, M) = (0)$ if $q > 0$ for line bundles M with non-negative degrees (by [1] and [8]). By duality, $R^q t_{L^*}(L) = (0)$ if $q < s$. (i) now follows from the Leray spectral sequence.

(ii) There is a P_f-equivariant map $R^s t_{L^*}(L) \to R^s t_{L^*}(L)_f$ obtained by restriction, hence there is such a map on the global sections. By the first part of the proof, $H^s(L)$ (respectively, $H^s(L_f)$) is $H^0(R^s t_{L^*}(L))$ (respectively, $H^0(R^s t_{L^*}(L_f))$).

LEMMA 2. *$H^0(P_L/B, L)$ (or $H^s(P_L/B, L)$) is indecomposable as a P_L-module. $H^0(P_L/B, L)$ contains a unique B-invariant line.*

Proof. If L has non-negative degrees, then $H^0(G/B, L)$ and $H^s(G/B, L)$ ($s = \dim G/B$) are indecomposable for any connected semi-simple linear algebraic group ([1], [5]). The second statement also follows from [1] and [5].

Note that there is a non-canonical isomorphism of $H^s(P_L/B, L)$ with $H^s(L_f)$, since f is non-canonically isomorphic to P_L/B. Also note that since $R^s f_{L^*}(L)$ is generated by its global sections, γ_f is surjective.

Proof of Theorem 1. Suppose $M_1 \oplus M_2 = H^q(L)$ is a non-trivial decomposition. For any fiber f, $\gamma_f(M_1) \cap \gamma_f(M_2)$ must be non-zero, hence must contain a P_f-irreducible submodule.

There are only finitely many such submodules. To see this, note that since L_f is a coherent sheaf, $H^q(L_f)$ is a finite-dimensional vector space on K. Hence there can be only a finite number of possible highest weights (with multiplicity), and so only a finite number of P_f-irreducible submodules. Since the P_f-submodules are all conjugates of the P_L-submodules of $H^q(L_F)$ (where F is the fiber stabilized by P_L), for a Zariski dense set X of fibers f the intersection $\gamma_f(M_1) \cap \gamma_f(M_2)$ contains the conjugates of a single P_L-irreducible submodule A.

There is a G-irreducible submodule of $H^q(L)$ contained in the inverse image of A under any γ_f. We have $H^q(L) = H^0(R^q t_{L^*}(L))$ by the proof of Lemma 1 and the Leray spectral sequence. By Serre duality,

$$R^q t_{L^*}(L) \otimes t_{L^*}(L^- \otimes \Omega) \to O_{G/P}$$

is a perfect pairing, where Ω is the sheaf of differentials on G/B relative to t_L. Hence

$$H^q(L) \cong \mathrm{Hom}_{O_{G/P}}\big(t_{L*}(L^{-1}\otimes\Omega), O_{G/P}\big).$$

Similarly,

$$H^q(L_f) \cong \mathrm{Hom}_{O_{G/P,y}}\big(t_{L*}(L^{-1}\otimes\Omega)_y, O_{G/P,y}\big),$$

where $y \in G/P$ is the point under the fiber f. A is the annihilator of the stalk of a homogeneous subsheaf M of $t_{L*}(L^{-1}\otimes\Omega)$; let $B \subseteq H^q(L)$ be the annihilator of that subsheaf.

To show $B \neq (0)$, note that it suffices to show the annihilator of M in $\mathbf{Hom}_{O_{G/P}}\big(t_*(L^{-1}\otimes\Omega), O_{G/P}\big)$ is generated by its global sections (B), since it has a non-zero stalk at y. Call this annihilator N. A sheaf S on G/P is generated by its global sections iff it is the quotient of a free sheaf ([12], p. 121):

$$0 \to K \to \oplus O_{G/P} \xrightarrow{a} S \to 0,$$

where a is given by global sections. Let

$$S = \mathbf{Hom}_{O_{G/P}}\big(t_*(L^{-1}\otimes\Omega), O_{G/P}\big).$$

Then $N = T/(K\cap T)$ for some subsheaf T of $\oplus O_{G/P}$. If T is free, then $B \neq (0)$. Suppose T is not free. Then $S/N \cong \oplus O_{G/P}/T$ is a sheaf whose rank is non-constant on G/P. G acts on the homogeneous sheaves S and N and G acts transitively on G/P, so this is impossible. Hence $B \neq (0)$.

Let C be a G-irreducible submodule of B. For any f, $\gamma_f(C) \subseteq \gamma_f(M_1)\cap \cap\gamma_f(M_2)$ by construction, hence $C \subseteq M_1+I_f$ and $C \subseteq M_2+I_f$ for any f in X, where I_f is the subspace of global sections of $R^q t_{L*}(L)$ vanishing on f. Hence

$$C \subseteq M_1 + \bigcap_{f\in X} I_f = M_1$$

(since X is dense) and, similarly, $C \subseteq M_2$. Hence $C \subseteq M_1\cap M_2$, which is a contradiction. This proves Theorem 1.

Proof of the Corollary. First assume $q = 0$. By the theorem cited in the proof of Lemma 1 (i), P_L must be B. Hence $s = 0$ and $H^s(P_L/B, L) = K$. Therefore, any submodule is indecomposable, in particular the image of γ_f. So the proof goes through as above. (This result is well known, see [1].)

If $q = 1$, then P_L/B must have dimension 1 by Lemma 1 (ii). (If it had dimension 0, then $H^1(L) = 0$ by the theorem cited above.) It is known in this case that $H^1(P_L/B, L)$ must have a unique irreducible submodule.

(The author does not know of a good reference for this, although the result follows from Section XXI of [11]. P_L/B is isomorphic to $SL(2)/B \cong \boldsymbol{P}^1$ in this case. $H^0(\boldsymbol{L})$ is the homogeneous polynomials of degree $d = \deg(\boldsymbol{L}|_f)$ in two variables X and Y. The maximal submodule is generated by all monomials $X^k Y^{d-k}$ such that $p^e \nmid d!/k!(d-k)!$, where e is the largest integer such that $p^e | d!/k_1!(d-k_1)!$ for some k_1 [11]. Use Serre duality.) This clearly implies that any submodule (including the image of γ_f) is indecomposable, so again the proof previously given works. H. H. Andersen has recently obtained this result by different means.

For $q = \dim G/B - 1$, $\dim G/B$ the Corollary follows by Serre duality.

In the theory of $H^*(G/B, \boldsymbol{L})$ in characteristic 0, it is true that one obtains an isomorphic copy of *any* irreducible G-module by choosing $\boldsymbol{L}$ suitably. It is natural to ask whether in characteristic p one can obtain an isomorphic copy of any given indecomposable G-module by taking a submodule of $H^*(G/B, \boldsymbol{L})$ for a suitable $\boldsymbol{L}$. Unfortunately, the answer is negative and an example follows.

Let $\mathrm{char}(K) = 2$. $SL(2)$ acts on $K[X_1, X_2, X_3, X_4]$ by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \\ a_{11}X_3 + a_{12}X_4 \\ a_{21}X_3 + a_{22}X_4 \end{pmatrix}.$$

Consider the homogeneous polynomials of degree 2. They form a module which is the direct sum of three indecomposable submodules, one of which is generated by X_1X_3, X_1X_4, X_2X_3, and X_2X_4. Let this module be M.

The weights of M are $2\overline{w}$, 0 (with multiplicity 2), and $-2\overline{w}$, where $\overline{w}$ is the fundamental weight. Since $H^0(SL(2)/B, \boldsymbol{L})$ and $H^1(SL(2)/B, \boldsymbol{L})$ have no weights of multiplicity greater than one, M cannot be a submodule (see [4] or note that $SL(2)/B = \boldsymbol{P}^1$, so $H^0(\boldsymbol{L}) = \mathrm{Sym}^{\deg \boldsymbol{L}}(X_1, X_2)$ and $X_1^k X_2^{j-k}$ has weight $(j-2k)\overline{w}$). (This reasoning fails in characteristic 0 because M is no longer indecomposable: it splits as the direct sum of modules generated by X_1X_3, $X_1X_4 + X_2X_3$, X_2X_4, and $X_1X_4 - X_2X_3$.)

If indecomposability is replaced by irreducibility, then the answer to the question is affirmative and easily proved:

THEOREM 2. *Suppose M is an irreducible G-module. Then M is isomorphic to a submodule of $H^0(G/B, \boldsymbol{L})$ for a suitable $\boldsymbol{L}$.*

Proof. Suppose M is infinitesimally irreducible. Hence its highest weight is of the form $\sum a_i \overline{w}_i$, $0 \leqslant a_i \leqslant p-1$ (see [2]). Hence if $\boldsymbol{L}$ is chosen to have degrees $a_1, \ldots, a_{n-1}$, then $H^0(\boldsymbol{L})$ contains an irreducible submodule which is isomorphic to M, since all degrees are less than p. Theorem 2 now follows from the Steinberg tensor product theorem [10].

REFERENCES

[1] L. Bai, C. Musili and C. S. Seshadri, *Cohomology of line bundles on G/B*, Annales Scientifiques de l'Ecole Normale Supérieure, 4-ième Série, 7 (1) (1974), p. 89-132.

[2] A. Borel, *Linear representations of semi-simple algebraic groups*, Proceedings of the American Mathematical Society, Symposium, Summer Institute, 1974.

[3] R. Bott, *Homogeneous vector bundles*, Annals of Mathematics 66 (1957), p. 203-248.

[4] R. Carter and E. Cline, *The submodule structure of Weyl modules for groups of type A_1*, Proceedings of the Conference on Finite Groups, Academic Press, 1976.

[5] C. Chevalley, *Classification des groupes de Lie algébriques*, Séminaire Chevalley, 1956-8, Exposés 15 et 16.

[6] W. L. Griffith, *Cohomology of flag varieties in characteristic p*, Thesis, Harvard University, 1975.

[7] — *Cohomology of line bundles in characteristic p* (to appear).

[8] G. Kempf, *Linear systems on homogeneous spaces* (to appear).

[9] — *Schubert methods with an application to algebraic curves*, Mathematisch Centrum, Amsterdam 1971.

[10] R. Steinberg, *Representations of algebraic groups*, Nagoya Mathematical Journal 22 (1963), p. 33-56.

[11] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, American Journal of Mathematics 1 (1878), p. 184-240.

[12] R. Hartshore, *Algebraic geometry*, Springer-Verlag, 1977.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI – ST. LOUIS
ST. LOUIS, MISSOURI

Reçu par la Rédaction le 2. 10. 1979