COLLOQUIUM MATHEMATICUM

VOL. XXI 1970 FASC. 1

FINITE SUBGROUPS OF LOCALLY COMPACT GROUPS

BY

S. ŚWIERCZKOWSKI (CANBERRA)

The aim of this note* is to solve a problem of J. Mycielski ([4], P 316, p. 137) and another one put forward by S. Hartman and J. Mycielski ([2], P 214, p. 169).

Mycielski conjectured in [4] that the infinite alternating group (i.e. the group of all even permutations of a countable infinite set) cannot be a subgroup (closed or not) of any locally compact connected group. This is so, by the following

THEOREM. *Every simple and periodic (not necessarily closed) subgroup K of a locally compact connected group G is finite.*

Proof. If G is a Lie group, the lemma is a consequence of the known fact that

($*$) every infinite, periodic subgroup of $GL(n, R)$ contains an infinite abelian normal subgroup ([1], 36. 17, p. 260, and 36. 11, p. 257).

Indeed, let K be simple, periodic and contained in G, and let $a\colon G \to GL(n, R)$ be the adjoint representation of G in its Lie algebra, where $n = \dim G$. Since $\operatorname{Ker} a \subset$ Centre G, it follows that $K \cap \operatorname{Ker} a = \{1\}$ and thus the composite $K \subset G \xrightarrow{a} GL(n, R)$ is a monomorphism. Thus, by ($*$), K cannot be infinite.

In the general case, when G is any locally compact connected group, we apply Yamabe's theorem ([5], Th. 5′, p. 364) by which G has arbitrarily small normal subgroups N such that G/N is a Lie group. Since $K \cap N$ is either $\{1\}$ or K, there must be some N for which $K \cap N = \{1\}$. For such N, the composite $K \subset G \to G/N$ is a monomorphism, moreover, G/N being a Lie group, the image of K in G/N is finite, by the above. Thus K is finite.

The problem put forward by J. Mycielski and S. Hartman in [2] was to prove the following

* Research supported by a grant from ZWO while the author was staying at the University of Leiden (Holland).

THEOREM. *Every periodic discrete subgroup of a connected locally compact group G is finite.*

Proof. Note first of all that, by the quoted theorem of Yamabe, the general case reduces to the one when G is a Lie group. In that latter case the lemma is a consequence of (*) and another known fact that

(**) every abelian discrete subgroup of a connected Lie group is finitely generated ([3], Th. 1′, p. 250).

Indeed, let Γ be a discrete, periodic subgroup of the locally compact, connected G. Denote the centre of G by Z. Then $\Gamma \cap Z$ is finite, by (**). Let $G_1 = G/(\Gamma \cap Z)$ and let Γ_1 be the image of Γ in G_1. Then Γ_1 is still discrete in G_1. Moreover, the adjoint representation of G_1 on its Lie algebra is faithful on Γ_1. Hence Γ_1 is isomorphic to a subgroup of $GL(n, R)$, where $n = \dim G_1$. If Γ_1 were infinite, then it would contain an infinite abelian subgroup, by (*). This group would then be a subgroup of G_1 which is discrete, periodic, infinite and abelian. But such a group cannot exist, by (**). Thus Γ_1 is finite and from $\Gamma_1 \cong \Gamma/(\Gamma \cap Z)$ we infer that Γ is finite.

REFERENCES

[1] C. W. Curtis and I. Reiner, *Representation theory of finite and associative algebras*, New York and London 1962.

[2] S. Hartman and J. Mycielski, *On the imbedding of topological groups into connected topological groups*, Colloquium Mathematicum 5 (1958), p. 167–169.

[3] G. D. Mostow, *On the fundamental group of a homogeneous space*, Annals of Mathematics 66 (1957), p. 249–255.

[4] J. Mycielski, *P 316*, Colloquium Mathematicum 8 (1961), p. 137.

[5] H. Yamabe, *A generalization of a theorem of Gleason*, Annals of Mathematics 58 (1953), p. 351–365.

AUSTRALIAN NATIONAL UNIVERSITY

Reçu par la Rédaction le 14. 1. 1969