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CANONICAL DIFFERENTIAL STRUCTURES OF REGULAR COVECTORS

BY

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0. Introduction. In [4] Sikorski generalized the notion of differentiable manifold by introducing the notion of differential space (see also [2]). The set of all tangent vectors and the set of all covectors tangent to a differential space M of the form $(\operatorname{set} M, \mathcal{F}(M))$, where $\mathcal{F}(M)$ denotes the differential structure of M, and $\operatorname{set} M$ its support, can be endowed with differential structures to form the differential space TM tangent to M and the differential space T^*M cotangent to M, respectively (see [3]). In [1] the space $^*TM$ of all regular tangent covectors is defined. The aim of this paper is to study some relations between $^*TM$ and T^*M. We give the negative answer to the question whether $^*TM$ can be regarded as a differential subspace of T^*M and we study when the answer to this question is positive. This makes a partial solution to the problem P 1261 from the paper [1].

1. Cotangent space and regular cotangent space. By $G_0(M, p)$, where M is a differential space and p is a point of M, we denote the real linear space of all germs at p of functions from $\mathcal{F}(M)$ whose values at p are zero. The quotient space $G_0(M, p)/G_0^2(M, p)$, where $G_0^2(M, p)$ is the linear space generated by the set $\{\alpha\beta\colon \alpha, \beta \in G_0(M, p)\}$, will be denoted by $^*T_pM$ and called the *space of all regular covectors tangent to M at p.* Its second dual will be denoted by T_p^*M and called the *space of all covectors tangent to M at p.* We can in a natural way regard $^*T_pM$ as a subspace of T_p^*M. More precisely, we have the natural mapping

$$i_p\colon {}^*T_pM \to T_p^*M$$

defined by the formula

$$\big(i_p(w)\big)(v) = v(w).$$

By T^*M we mean the differential space with

$$\operatorname{set} T^*M = \bigcup_{p \in \operatorname{set} M} T_p^*M$$

and $\mathscr{F}(T^* M)$ generated (see [2], [5] and [6]) by the set

$$\{\alpha \circ \pi;\ \alpha \in \mathscr{F}(M)\} \cup \{\tilde{X};\ X \in \mathfrak{X}(M)\},$$

where π: set $T^* M \to$ set M, $\pi(w) = p$ for $w \in T_p^* M$, $\mathfrak{X}(M)$ denotes the set of all smooth vector fields tangent to M, $\tilde{X}$: set $T^* M \to \boldsymbol{R}$ is defined by the formula

$$\tilde{X}(w) = w\big(X(\pi(w))\big).$$

Now we define the space $^*TM$. Denote by $d\alpha_p \in T_p^* M$, where $\alpha \in \mathscr{F}(M)$, p is a point of M, the element $[\alpha - \alpha(p), p] + G_0^2(M, p)$, where $[\beta, p]$ denotes the germ of β at p. We put

$$\operatorname{set} {}^*TM = \bigcup_{p \in \operatorname{set} M} {}^*T_p M$$

and

$$\mathscr{F}(^*TM) = \{\xi \in \boldsymbol{R}^{\operatorname{set}^*TM};\ \text{for any finite } B \subset \mathscr{F}(M)$$
$$\text{there is } \big(\boldsymbol{R}^B \times \operatorname{set} M \ni (c, p) \mapsto \xi\big(\sum_{\alpha \in B} c_\alpha\, d\alpha_p\big) \in \boldsymbol{R}\big) \in \mathscr{F}(\boldsymbol{R}^B \times M)\big\},$$

where $\mathscr{F}(\boldsymbol{R}^B \times M) = C^\infty(\boldsymbol{R}^B) \times \mathscr{F}(M)$ is the standard product differential structure in $\boldsymbol{R}^B \times \operatorname{set} M$. The projection

$$^*\pi\colon \operatorname{set} {}^*TM \to \operatorname{set} M,$$

where $^*\pi(d\alpha_p) = p$ for $p \in \operatorname{set} M$ and $\alpha \in \mathscr{F}(M)$, is a smooth function from $^*TM$ onto M. Moreover, for any smooth vector field X on M we have $X \in \mathscr{F}(^*TM)$ (see [1]). Put

$$i(w) = i_{^*\pi(w)}(w) \quad \text{for } w \in \operatorname{set} {}^*TM.$$

Then $\tilde{X} \circ i = X$ and $\pi \circ i = {}^*\pi$.

We can write the above remarks in the following form:

THEOREM 1. *The mapping* i: $^*TM \to T^* M$ *is smooth for any differential space* M.

We will study when $^*TM$ can be regarded as a differential subspace of $T^* M$, more precisely, when the mapping

$$i^{-1}\colon {}^*T^* M \to {}^*TM$$

is smooth, where $^*T^* M$ denotes the space $\big(i(\operatorname{set} {}^*TM), \mathscr{F}(T^* M)_{i(\operatorname{set}^*TM)}\big)$.

2. The question of smoothness of i^{-1}. Andrzejczak [1] proved that in the case where M is a differentiable manifold the mapping i is a diffeomorphism. We generalize this result.

We say that M has a *finite local basis at a point* p if there are a neighbourhood U of p and vector fields $X_1, X_2, \ldots, X_n$ defined on M such that for any point q of U the vectors $X_1(q), X_2(q), \ldots, X_n(q)$ form a basis of the space $T_q M$ tangent to M at q.

THEOREM 2. *If M has a finite local basis at any point, then i is a diffeomorphism.*

Proof. We are to prove that for any $\xi \in \dot{\mathscr{F}}(^*TM)$ and any point w of $^*TM$ there are smooth vector fields $Y_1, Y_2, \ldots, Y_l$ on M, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathscr{F}(M)$, $f \in C^\infty(\boldsymbol{R}^{k+l}, \boldsymbol{R})$ and a neighbourhood U_0 of w in $^*TM$ such that

$$\xi(v) = f\big(Y_1(v), Y_2(v), \ldots, Y_l(v), \alpha_1 \circ {}^*\pi(v), \alpha_2 \circ {}^*\pi(v), \ldots, \alpha_k \circ {}^*\pi(v)\big)$$

for $v \in U_0$.

Consider any ξ and w as above. Put $p = {}^*\pi(w)$. Let $\beta^1, \beta^2, \ldots, \beta^n$ be smooth real functions on M such that $d\beta^1_p, d\beta^2_p, \ldots, d\beta^n_p$ form a basis of $^*T_p M$. Let $X_1, X_2, \ldots, X_n$ be any local basis of M at p on a neighbourhood U_1 of p. For any $i, j \in \{1, 2, \ldots, n\}$ we have $X_i(\beta^j) \in \mathscr{F}(M)$. Hence $D = \det[X_i(\beta^j);\ i, j \leqslant n]$ is a smooth function on M. Since $D(p) \neq 0$, there is a neighbourhood $U_2 \subset U_1$ of p such that $D(q) \neq 0$ for $q \in U_2$. Let $A = [a^i_j;\ i, j \leqslant n]$ be the inverse matrix to $B = [X_i(\beta^j);\ i, j \leqslant n]$. The functions a^i_j are correctly defined on U_2. For any i, j the function $a^i_j \cdot \det B$ is a polynomial function of the coefficients of the matrix B. Hence a^i_j is smooth on the set U_2. For $1 \leqslant l \leqslant n$ let Y_l be a smooth vector field on M equal to $\sum_{i=1}^{n} a^i_l X_i$ on a neighbourhood of p. Then there is a neighbourhood U_3 of p such that the fields $Y_1, Y_2, \ldots, Y_n$ form a local basis of M at p on U_3 and, moreover, for $j, l \leqslant n$ and $q \in U_3$ we have

$$Y_l(q)(\beta^j) = \sum_{i=1}^{n} a^i_l X_i(q)(\beta^j) = \delta^j_l.$$

By the definition of the differential structure of $^*TM$, the mapping

$$\varphi = \Big(\boldsymbol{R}^n \times \operatorname{set} M \ni (c_1, c_2, \ldots, c_n, q) \mapsto \xi\big(\sum_{i=1}^{n} c_i\, d\beta^i_q\big) \in \boldsymbol{R}\Big)$$

is a smooth mapping of $\boldsymbol{R}^n \times M$ into $\boldsymbol{R}$. By the definition of the product of differential spaces, there is a mapping Ψ of the form

$$(c_1, c_2, \ldots, c_n, q) \mapsto f\big(c_1, c_2, \ldots, c_n, \alpha_1(q), \alpha_2(q), \ldots, \alpha_k(q)\big),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathscr{F}(M)$ and $f \in C^\infty(\boldsymbol{R}^{k+n}, \boldsymbol{R})$, such that $\varphi | V \times U = \Psi | V \times U$, where V is a neighbourhood of $\big(Y_1(w), Y_2(w), \ldots, Y_n(w)\big)$ in $\boldsymbol{R}^n$ and $U \subset U_3$ is a neighbourhood of $p = {}^*\pi(w)$.

We show that the set

$$U_0 = \{v \in \operatorname{set} {}^*TM;\ {}^*\pi(v) \in U \text{ and } \big(Y_1(v), Y_2(v), \ldots, Y_n(v)\big) \in V\}$$

is open in $^*TM$. Since $^*\pi\colon {}^*TM \to M$ is smooth, the set $^*\pi^{-1}(U)$ is open in $^*TM$. Similarly, the mapping

$$\gamma = \left(\text{set}\, {}^*TM \ni v \mapsto (Y_1(v), Y_2(v), \ldots, Y_n(v)) \in \boldsymbol{R}^n\right)$$

is smooth, which shows that $\gamma^{-1}(V)$ is open. Hence

$$U_0 = {}^*\pi^{-1}(U) \cap \gamma^{-1}(V)$$

is open.

Now, it suffices to observe that for $v \in U_0$ we have

$$f\left(Y_1(v), Y_2(v), \ldots, Y_n(v), \alpha_1({}^*\pi(v)), \alpha_2({}^*\pi(v)), \ldots, \alpha_k({}^*\pi(v))\right)$$
$$= \xi\left(\sum_{i=1}^{n} Y_i(v)\, d\beta^i_{{}^*\pi(v)}\right) = \xi(v),$$

which completes the proof.

The following example shows that Theorem 2 cannot be generalized for spaces with finite dimension of the tangent space at any point.

Example 1. Put

$$\text{set}\, M = \{(n, x);\ n \in \boldsymbol{N} \cup \{0\},\ x \in \boldsymbol{R} \text{ and } x > 0\} \cup \{(0, 0)\}.$$

Let $\mathscr{F}(M)$ be the set of all real functions α defined on set M such that for any n the function $x \mapsto \alpha(n, x)$ is smooth, i.e., can be extended to a C^∞-function on $\boldsymbol{R}$ and for any $n \neq 0$ there is $\varepsilon > 0$ such that $\alpha(n, \delta) = \alpha(0, 0)$ for $\delta < \varepsilon$.

We verify directly that we have defined a differential space M. A point p of M will also be written in the form (p_1, p_2).

We prove the following

Lemma (a version of the Hilbert theorem on basis). *Let $P_1, P_2, \ldots$ be any polynomials in l variables over the field of reals. Then we find a natural N such that for $m > N$ we have $P_m \in I(P_1, P_2, \ldots, P_N)$, where $I(P_1, P_2, \ldots, P_N)$ denotes the ideal generated by polynomials $P_1, P_2, \ldots, P_N$ in the ring $\boldsymbol{R}[x_1, x_2, \ldots, x_l]$.*

Proof. The ideal $I = I(\{P_n;\ n \in \boldsymbol{N}\})$ is the union

$$\bigcup_{n=1}^{\infty} I(P_1, P_2, \ldots, P_n).$$

Since I is finitely generated, we find N such that

$$I = I(P_1, P_2, \ldots, P_N).$$

Consequently, we have $P_m \in I(P_1, P_2, \ldots, P_N)$ for any natural m.

Now we can prove the following

Theorem 3. *The differential space M from Example 1 has finite dimension* *of the tangent space at any point. However, the mapping i is not a diffeomorphism.*

Proof. Since M is locally diffeomorphic to $\boldsymbol{R}$ in a neighbourhood of any point $p \in \operatorname{set} M \setminus \{(0, 0)\}$, we have $\dim {}^*T_p M = 1$ for $p \neq (0, 0)$. Moreover, we have $\dim {}^*T_{(0,0)} M = 1$, because for $\alpha \in \mathscr{F}(M)$ such that

$$\lim_{h \to 0^+} \frac{\alpha(0, h)}{h} = 0$$

we have $d\alpha_{(0,0)} = 0$. Thus the tangent space has dimension 1 at any point. Neighbourhoods of (0, 0) are sets $U \subset \operatorname{set} M$ such that for any nonnegative integer n the set $\{x;\ (n, x) \in U\} \cup \{0\}$ is a neighbourhood of 0 in $\boldsymbol{R}^+ \cup \{0\}$.

Denote by $\overline{(p, \alpha)}$, where $p \in \operatorname{set} M$, $\alpha \in \mathscr{F}(M)$, the derivative of α at p, i.e., the limit

$$\lim_{h \to 0} \frac{\alpha(p_1, p_2 + h) - \alpha(p_1, p_2)}{h}$$

in the case $(p_1, p_2) \neq (0, 0)$ and the same limit with $h \to 0$ replaced by $h \to 0^+$ in the case $p_1 = p_2 = 0$. This operation can be transported to elements of ${}^*TM$ by setting $\overline{d\alpha_p} = \overline{(p, \alpha)}$. The element $d\alpha_p$ will be denoted by $(p;\ \overline{d\alpha_p})$.

Put for $v \in \operatorname{set} {}^*TM$

$$\xi(v) = \begin{cases} (\bar{v})^{({}^*\pi(v))_1} & \text{for } ({}^*\pi(v))_1 \neq 0, \\ 0 & \text{for } ({}^*\pi(v))_1 = 0, \end{cases}$$

where $({}^*\pi(v))_1$ stands for the first coordinate of ${}^*\pi(v)$. We will show that $\xi \in \mathscr{F}({}^*TM)$. Consider any $\alpha^1, \alpha^2, \ldots, \alpha^k \in \mathscr{F}(M)$. We are to show that the mapping Φ of the form

$$\boldsymbol{R}^k \times \operatorname{set} M \ni (c_1, c_2, \ldots, c_k, p) \mapsto \xi\Big(\sum_{i=1}^{k} c_i\, d\alpha_p^i\Big)$$

is in $\mathscr{F}(\boldsymbol{R}^k \times M)$. It suffices to verify that any point p of M has a neighbourhood U such that $\Phi | \boldsymbol{R}^k \times U$ is smooth. For p with $p_1 \neq 0$ this statement holds, because for q from the set $\boldsymbol{N} \times \boldsymbol{R}^+$ we have

$$\xi\Big(\sum_{i=1}^{k} c_i\, d\alpha_q^i\Big) = \Big(\sum_{i=1}^{k} c_i\, \overline{d\alpha_q^i}\Big)^{q_1}$$

and the mapping $\operatorname{set} M \ni q \mapsto \overline{d\alpha_q^i}$ is smooth on a neighbourhood of p. For p with $p_1 = 0$ we find a neighbourhood U of p such that $\Phi | \boldsymbol{R}^k \times U$ is a zero mapping.

Now we will show that there is no function Ψ: $\operatorname{set} {}^*TM \to \boldsymbol{R}$ of the form

$$v \mapsto f\big(X_1(v), X_2(v), \ldots, X_l(v), \beta_1({}^*\pi(v)), \beta_2({}^*\pi(v)), \ldots, \beta_j({}^*\pi(v))\big),$$

where $f \in C^\infty(\mathbf{R}^{l+j}, \mathbf{R})$, $X_1, X_2, \ldots, X_l$ are vector fields on M, and $\beta_1, \beta_2, \ldots, \beta_j \in \mathscr{F}(M)$, such that $\Psi|V = \xi|V$ for some open neighbourhood V of a zero covector at $(0, 0)$. Let us consider any f, $X_1, X_2, \ldots, X_l$, $\beta_1, \beta_2, \ldots, \beta_k$, Ψ, V as above and suppose that $\Psi|V = \xi|V$. Since V is open in $^*TM$ and for any $q \in \operatorname{set} M$ the mappings

$$\operatorname{set} M \ni p \mapsto (p; 0) \quad \text{and} \quad \mathbf{R} \ni a \mapsto (q; a)$$

are smooth, we find that the set $\{p \in \operatorname{set} M; (p; 0) \in V\}$ is open in M and for any q of M the set $\{a \in \mathbf{R}; (q; a) \in V\}$ is open in $\mathbf{R}$. For $n = 1, 2, \ldots$ let x_n denote a positive real such that for $0 \leqslant x \leqslant x_n$ we have $((n, x); 0) \in V$ and

$$\beta_1(n, x) = \beta_1(0, 0), \quad \beta_2(n, x) = \beta_2(0, 0), \quad \ldots, \quad \beta_j(n, x) = \beta_j(0, 0).$$

Putting, for $i = 1, 2, \ldots, l$ and $n = 1, 2, \ldots$,

$$D_i^n = X_i((n, x_n); 1),$$

we have

$$t^n = \Psi((n, x_n); t) = F(D_1^n t, D_2^n t, \ldots, D_l^n t) \quad \text{for } ((n, x_n); t) \in V,$$

where

$$F(t_1, t_2, \ldots, t_l) = f(t_1, t_2, \ldots, t_l, \beta_1(0, 0), \beta_2(0, 0), \ldots, \beta_j(0, 0)).$$

Since $\{t \in \mathbf{R}; ((n, x_n); t) \in V\}$ is an open neighbourhood of 0 for any n, we find that for any natural m and n we have

$$F^{(m)}(D_1^n, D_2^n, \ldots, D_l^n) = \delta_m^n \cdot n!,$$

where $F^{(m)}(t_1, t_2, \ldots, t_l)$ denotes the m-th derivative of F at 0 in the direction $(t_1, t_2, \ldots, t_l)$. By the Taylor formula, $F^{(m)}$ is a polynomial of variables $t_1, t_2, \ldots, t_l$. By the Lemma we find an N such that for any $t_1, t_2, \ldots, t_l \in \mathbf{R}$ the condition

$$F^{(n)}(t_1, t_2, \ldots, t_l) = 0 \quad \text{for } n \leqslant N$$

yields

$$F^{(m)}(t_1, t_2, \ldots, t_l) = 0$$

for any natural m. But we have

$$F^{(n)}(D_1^{N+1}, D_2^{N+1}, \ldots, D_l^{N+1}) = 0 \quad \text{for } n \leqslant N$$

and

$$F^{(N+1)}(D_1^{N+1}, D_2^{N+1}, \ldots, D_l^{N+1}) \neq 0.$$

This shows that $\xi \circ i^{-1} \notin \mathscr{F}(T^* M)$. Therefore i^{-1} is not smooth.

The following question seems to be interesting:

PROBLEM (**P 1344**). Is the mapping i a diffeomorphism in the case where M is a differential subspace of a differential space with a finite local basis at any point? What is the answer when M is a differential subspace of a Euclidean space? The autor does not know the answers.

Now we see that, in general, ${}^*TM$ cannot be regarded as a differential subspace of $T^* M$. However, Andrzejczak [1] proved that, in the case where ${}^*T_p M$ is of finite dimension, i_p is a diffeomorphism. In the last part of the paper we show that this statement does not hold in general.

3. The case where ${}^*T_p M$ is of infinite dimension. Now we study the question of smoothness of addition of vectors and covectors tangent to a differential space. First, we prove a general theorem.

THEOREM 4. *Let N be a differential space whose support is a commutative group* $(\operatorname{set} N, +)$. *Let us suppose that the differential structure of N is generated by a set C of homomorphisms of* $(\operatorname{set} N, +)$ *into* $(\boldsymbol{R}, +)$. *Then the addition* $+\colon \operatorname{set} N \times \operatorname{set} N \to \operatorname{set} N$ *is a smooth mapping from $N \times N$ into N.*

Proof. We are to prove that for any $\alpha \in C$ the mapping

$$\operatorname{set}(N \times N) \ni (q, r) \mapsto \alpha(q+r)$$

is in $\mathscr{F}(N \times N)$. It suffices to write this mapping in the form $\alpha \circ \pi_1 + \alpha \circ \pi_2$, where π_1 and π_2 are the respective projections of $N \times N$ onto N.

COROLLARY. *For any differential space M the addition of covectors tangent to M at a point p is smooth with respect to the structure $\mathscr{F}(T^* M)_{T_p^* M}$ on $T_p^* M$.*

We have similar conclusions for addition in the tangent space $T_p M$ and the space ${}^*T_p^* M$ of all regular covectors tangent to M at p with the structure induced from $T^* M$.

We give the example of a space M such that the addition of regular covectors is not smooth with respect to the structure induced from ${}^*TM$! Consequently, we show that i_p is not, in general, a diffeomorphism.

EXAMPLE 2. Let H be a separable Hilbert space. We regard it as the differential space. Denote this space by M. Let $\operatorname{set} M$ be the set of all points of H. By the differential structure on M we mean the set of all real C^∞-functions defined on H.

THEOREM 5. *The addition of regular covectors is not smooth in the space* $({}^*T_0 M, \mathscr{F}({}^*TM)_{{}^*T_0 M})$.

Proof. For any $\alpha \in \mathscr{F}(M)$ denote by $D\alpha$ the derivative function of α. $D\alpha$ is a C^∞-mapping from H into H. It is easy to see that for $\alpha, \beta \in \mathscr{F}(M)$ and $p \in \operatorname{set} M$ the condition $d\alpha_p = d\beta_p$ yields $D\alpha(p) = D\beta(p)$. Setting $\xi(d\alpha_p)$

$= \|D\alpha(p)\|^2$ we have a real-valued function ξ on set $^*TM$. We show that $\xi \in \mathscr{F}(^*TM)$. Let us consider any $\alpha^1, \alpha^2, \ldots, \alpha^k$ from $\mathscr{F}(M)$. We have

$$\xi\Big(\sum_{i=1}^{k} c_i\, d\alpha_p^i\Big) = \Big\|\sum_{i=1}^{k} c_i\, D\alpha^i(p)\Big\|^2 = \sum_{i,j=1}^{k} c_i c_j \langle D\alpha^i(p), D\alpha^j(p)\rangle.$$

Since for $1 \leqslant i, j \leqslant k$ the mapping $p \mapsto \langle D\alpha^i(p), D\alpha^j(p)\rangle$ is smooth, we have the smooth mapping

$$\boldsymbol{R}^k \times \operatorname{set} M \ni (c_1, c_2, \ldots, c_k, p) \mapsto \xi\Big(\sum_{i=1}^{k} c_i\, d\alpha_p^i\Big) \in \boldsymbol{R}$$

of $\boldsymbol{R}^k \times M$. Hence $\xi \in \mathscr{F}(^*TM)$ and, consequently, $\xi | {}^*T_0 M \in \mathscr{F}(^*TM)_{^*T_0 M}$. To prove that the addition in $^*T_0 M$ is not smooth it suffices to show that the mapping

$$^*T_0 M \times {}^*T_0 M \ni (v, w) \mapsto \xi(v+w)$$

is not in $\mathscr{F}(^*TM)_{^*T_0 M} \times \mathscr{F}(^*TM)_{^*T_0 M}$.

Let us suppose that there are $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_m$ in $\mathscr{F}(^*TM)_{^*T_0 M}$, $f \in C^\infty(\boldsymbol{R}^{m+n}, \boldsymbol{R})$ and neighbourhoods U and V of 0 in $^*T_0 M$ such that for $(v, w) \in U \times V$ we have

$$\xi(v+w) = f\big(\varphi_1(v), \varphi_2(v), \ldots, \varphi_n(v), \psi_1(w), \psi_2(w), \ldots, \psi_m(w)\big).$$

We show that the above supposition leads to a contradiction. Consider any orthonormal vectors $x^1, x^2, \ldots, x^{n+1}$ of H. Let $\alpha^i(p) = \langle x^i, p\rangle$ for $i = 1, 2, \ldots, n+1$ and

$$h(c) = \sum_{i=1}^{n+1} c_i\, d\alpha_0^i \quad \text{for } c = (c_1, c_2, \ldots, c_{n+1}) \in \boldsymbol{R}^{n+1}.$$

The mapping $h\colon \boldsymbol{R}^{n+1} \to {}^*T_0 M$ is smooth (see [1]). Moreover, for any $c \in \boldsymbol{R}^{n+1}$ we have

$$\xi\big(h(c)\big) = \|c\|^2 = \sum_{i=1}^{n+1} c_i^2.$$

We have the smooth mapping F of the form

$$\boldsymbol{R}^{n+1} \times \boldsymbol{R}^{n+1} \ni (c, e) \mapsto \|c+e\|^2,$$

which is identical to

$$(c, e) \mapsto f\big(\varphi_1(h(c)), \varphi_2(h(c)), \ldots, \varphi_n(h(c)), \psi_1(h(e)), \psi_2(h(e)), \ldots, \psi_m(h(e))\big)$$

on the open set $h^{-1}(U) \times h^{-1}(V)$ containing $(0, 0)$. F is the function of $2n+2$ real variables, defined by the formula

$$F(x_1, x_2, \ldots, x_{2n+2}) = \sum_{i=1}^{n+1} (x_i + x_{n+i+1})^2.$$

Moreover, the functions

$$\Phi_1 = \varphi_1 \circ h, \quad \Phi_2 = \varphi_2 \circ h, \quad \ldots, \quad \Phi_n = \varphi_n \circ h,$$
$$\Psi_1 = \psi_1 \circ h, \quad \Psi_2 = \psi_2 \circ h, \quad \ldots, \quad \Psi_m = \psi_m \circ h$$

are in $C^\infty(\boldsymbol{R}^{n+1}, \boldsymbol{R})$. Consider the matrix

$$A = [\partial_i \partial_{n+j+1} F(0);\ 1 \leqslant i, j \leqslant n+1]$$

of partial derivatives of F at 0. We have

$$A = [2\delta_{ij};\ 1 \leqslant i, j \leqslant n+1].$$

Hence $\operatorname{rank} A = n+1$.

On the other hand, we have

$$\partial_i \partial_{n+j+1} F(0) = \sum_{k=1}^{n} \sum_{l=1}^{m} (\partial_k \partial_{n+l} f)(\Phi_1(0), \Phi_2(0), \ldots, \Phi_n(0),$$
$$\Psi_1(0), \Psi_2(0), \ldots, \Psi_m(0))\, \partial_i \Phi_k(0)\, \partial_j \Psi_l(0) = \sum_{k=1}^{n} A_k^{ij},$$

where

$$A_k^{ij} = \partial_i \Phi_k(0) \sum_{l=1}^{m} \partial_j \Psi_l(0) (\partial_k \partial_{n+l} f)(\Phi_1(0), \Phi_2(0), \ldots, \Phi_n(0),$$
$$\Psi_1(0), \Psi_2(0), \ldots, \Psi_m(0)).$$

Hence

$$A = \sum_{k=1}^{n} A_k,$$

where $A_k = [A_k^{ij};\ 1 \leqslant i, j \leqslant n+1]$ is a matrix of rank 1 for any k. Then $\operatorname{rank} A \leqslant n$, which contradicts the equality $\operatorname{rank} A = n+1$. The proof is complete.

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