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ON A GENERALIZATION OF THE NOTION OF A SEMI-GROUP

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0. In the present paper we consider quasi-algebras of the form $\langle A, \circ\rangle$, where A is a set and $\circ$ is a binary partial operation in A (see [1]). These quasi-algebras are of more general type than semi-groups or abstract categories, if elements of an abstract category are elements of some set (see [2]).

Theorem 1 gives an axiomatic characterization of quasi-algebras of functions. The domains of these functions are assumed to be non-empty sets and the partial operation a superposition of functions. Section 2 concerns the possibility of an extension of quasi-algebras, which have the form $\langle A, \circ\rangle$ and satisfy axioms (1.0)-(1.3), to semi-groups. From a possibility of an extension of the quasi-algebra $\langle A, \circ\rangle$ to a semi-group it follows that $\langle A, \circ\rangle$ satisfies axiom (1.0), but it does not follow that $\langle A, \circ\rangle$ satisfies axioms (1.1)-(1.3).

For a given function f we denote by D_f the domain of f. The inverse image of A by f is denoted by $f^{-1}[A]$. For a given set $A \subset D_f$ we denote the image of A by $f[A]$. In particular, any partial operation is a function and by *domain* of partial operation we shall mean a domain of that function.

1. Let C be an arbitrary set of functions, the domains of which are non-empty sets. Every pair $\langle g, f\rangle$ of functions of C, which satisfy the condition

$$f[D_f] \subset D_g, \tag{1}$$

determines the function $g\circ_C f$ defined for $x \in D_f$ by the formula

$$(g\circ_C f)(x) = g\big(f(x)\big). \tag{2}$$

The pair $\langle C, \circ_C\rangle$ is a quasi-algebra (see [1]) if and only if for every pair $\langle g, f\rangle$ of functions of C which satisfy (1) the function $g\circ_C f$ defined on the set D_f by (2) also belongs to C, and its domain is the set

$$D_{\circ_C} = \{\langle g, f\rangle : g, f \in C \wedge f[D_f] \subset D_g\}. \tag{3}$$

Any quasi-algebra of this type will be called a *quasi-semi-group of functions*.

We shall consider quasi-algebras $\langle A, \circ\rangle$, which satisfy the following conditions:

$$(1.0)\qquad \bigwedge_{x,y,z}\big(\langle x, y\rangle, \langle x\circ y, z\rangle, \langle y, z\rangle, \langle x, y\circ z\rangle \in D_\circ \Rightarrow (x\circ y)\circ z = x\circ(y\circ z)\big),$$

$$(1.1)\qquad \bigwedge_{x,y,z}(\langle x, y\rangle, \langle y, z\rangle \in D_\circ \Rightarrow \langle x\circ y, z\rangle \in D_\circ),$$

$$(1.2)\qquad \bigwedge_{x,y,z}(\langle x, y\rangle, \langle y, z\rangle \in D_\circ \Rightarrow \langle x, y\circ z\rangle \in D_\circ),$$

$$(1.3)\qquad \bigwedge_{x,y,z}(\langle x, y\rangle, \langle x\circ y, z\rangle \in D_\circ \Rightarrow \langle y, z\rangle \in D_\circ).$$

Any quasi-algebra satisfying these conditions will be called a *quasi-semi-group*.

THEOREM 1. *A quasi-algebra $\langle A, \circ\rangle$ is a quasi-semi-group if and only if there exists a quasi-semi-group $\langle C, \circ_C\rangle$ of functions isomorphic with $\langle A, \circ\rangle$.*

Proof. Suppose that $\langle A, \circ\rangle$ satisfies axioms (1.0)-(1.3). Define

$$E = \{e\colon e\in A \wedge \bigwedge_x(\langle x, e\rangle \in D_\circ \Rightarrow x\circ e = x)\},$$

$$R = \{x\colon x\in A \wedge \bigwedge_{e\in E}(\langle x, e\rangle \notin D_\circ)\}$$

and

$$(4)\qquad D_\circ(x) = \{y\colon \langle x, y\rangle \in D_\circ\}\quad \text{for}\quad x\in R.$$

Evidently, there exists a function a which maps the set R onto some T disjoint with A and such that

$$\bigwedge_{u,v\in R}\big(a(u) = a(v) \Leftrightarrow D_\circ(u) = D_\circ(v)\big).$$

Denote by B the set $A \cup T$ and put

$$S_1 = \{\langle x, a(x)\rangle\colon x\in R\},\qquad S_2 = \{\langle a(t), y\rangle\colon t\in R \wedge y\in D_\circ(t)\},$$

$$S_3 = \{\langle a(t), a(t)\rangle\colon t\in R\}.$$

Now define a partial operation $\odot$ in the following way:

$$x\odot y = \begin{cases} x\circ y & \text{for}\quad \langle x, y\rangle \in D_\circ, \\ y & \text{for}\quad \langle x, y\rangle \in S_2, \\ x & \text{for}\quad \langle x, y\rangle \in S_1 \cup S_3. \end{cases}$$

The domain of $\odot$ is the set $D_\odot = D_\circ \cap S_1 \cup S_2 \cup S_3$. It requires an easy verification that the ordered pair $\langle B, \odot\rangle$ is a quasi-algebra for which conditions (1.0)-(1.3) hold. Moreover, $\langle A, \circ\rangle$ is a subquasi-algebra of $\langle B, \odot\rangle$, and it fulfills the condition

$$(1.4)\qquad \bigwedge_{x\in B}\bigvee_e\big(\langle x, e\rangle \in D_\odot \wedge \bigwedge_u(\langle u, e\rangle \in D_\odot \Rightarrow u\odot e = u)\big).$$

In order to show it, consider an arbitrary $e \in E \cup T$ and an arbitrary u such that $\langle u, e\rangle \in D_{\odot}$. If $\langle u, e\rangle \in D_{\circ}$, then $e \in E$, because $A \cap T = \emptyset$. Hence $u \odot e = u \circ e = u$. If $\langle u, e\rangle \in S_1 \cup S_3$, then $u \odot e = u$. And $\langle u, e\rangle \in S_2$ is impossible, because there would exist $t \in R$ such that $u = a(t)$ and $e \in D_{\circ}(t)$, whence $e \notin T$. Thus $e \in T$ and $\langle t, e\rangle \in D_{\circ}$, which would contradict the definition of R. Consequently, in all cases we have $u \odot e = u$.

Consider now an arbitrary $x \in B$. If $x \in A \setminus R$, then there exists $e \in E$ such that $\langle x, e\rangle \in D_{\circ}$. If $x \in R$, then $\langle x, a(x)\rangle \in D_{\odot}$ and $a(x) \in T$. If $x \in T$, then $\langle x, x\rangle \in D_{\odot}$. Hence the quasi-algebra $\langle B, \odot\rangle$ satisfies condition (1.4).

Consider the function f which maps the set B into the set $\bigcup \{B^X \colon \emptyset \neq X \subset B \times B\}$ and is defined as

$$\big(f(u)\big)(x) = u \odot x \quad \text{for} \quad x \in D_{\odot}(u), \tag{5}$$

where $D_{\odot}(u) = \{x \colon \langle u, x\rangle \in D_{\odot}\}$. Put $H = f[B]$. Hence the set H is a set of functions. We prove that the ordered pair $\langle H, \circ_H\rangle$ is a quasi-algebra isomorphic with the quasi-algebra $\langle B, \odot\rangle$.

Let $\langle u, v\rangle \in D_{\odot}$. Consider an arbitrary $x \in D_{\odot}(v)$. It follows from axiom (1.2) that $\langle u, v \odot x\rangle \in D_{\odot}$. In other words, $\big(f(v)\big)(x) \in D_{\odot}(u)$. Consequently, we obtain

$$\big(f(v)\big)[D_{\odot}(v)] \subset D_{\odot}(u).$$

Suppose now that $u, v \in B$ and that $\big(f(v)\big)[D_{\odot}(v)] \subset D_{\odot}(u)$. It follows from (1.4) that there exists an $e \in D_{\odot}(v)$ and that $v = v \odot e = \big(f(v)\big)(e)$. Hence $v \in D_{\odot}(u)$. In other words, $\langle u, v\rangle \in D_{\odot}$. Now we remark that if $\langle u, v\rangle \in D_{\odot}$, then

$$f(u \odot v) = f(u) \circ_H f(v). \tag{6}$$

Indeed, it follows from (1.1) and (1.3) that $D_{\odot}(u \odot v) = D_{\odot}(v)$. Condition (1.0) implies identity (6). It follows from (1.4) that if $u, v \in B$ and $f(u) = f(v)$, then $u = v$. Therefore the function f is an isomorphism of the quasi-algebra $\langle B, \odot\rangle$ onto the quasi-algebra $\langle H, \circ_H\rangle$.

Let $C = f[A]$. Then the quasi-algebras $\langle A, \circ\rangle$ and $\langle C, \circ_C\rangle$ are isomorphic, and the quasi-algebra $\langle C, \circ_C\rangle$ is a quasi-semi-group of functions. Since the quasi-algebra $\langle C, \circ_C\rangle$ satisfies conditions (1.0)-(1.3), the quasi-algebra $\langle A, \circ\rangle$ satisfies them too. Thus the proof is complete.

2. Consider an arbitrary set H of functions. Any ordered pair $\langle g, f\rangle$ of functions belonging to H determines the function $g \circ'_H f$ defined for $x \in f^{-1}[D_g]$ by the formula

$$(g \circ'_H f)(x) = g\big(f(x)\big).$$

If the set H satisfies the condition

$$\bigwedge_{g,f} (g, f \in H \Rightarrow g \circ'_H f \in H), \tag{2.1}$$

then $\langle H, \circ'_H\rangle$ is a semi-group.

THEOREM 2. *If $\langle A, \circ \rangle$ is a quasi-semi-group, then there exists a semi-group that is an extension of $\langle A, \circ \rangle$.*

Proof. Suppose that the quasi-algebra $\langle A, \circ \rangle$ satisfies conditions (1.0)-(1.3). Theorem 1 yields existence of a quasi-semi-group $\langle C, \circ_C \rangle$ of functions which is isomorphic with the quasi-algebra $\langle A, \circ \rangle$. Then among those sets H (elements of H are funçtions), which contain C and satisfy (2.1), there exists the smallest one. Let $\bar{C}$ denote this set. Hence the pair $\langle \bar{C}, \circ'_{\bar{C}} \rangle$ is a semi-group. If a pair $\langle g, f \rangle$ of functions belonging to C satisfies (1), then $g \circ_C f = g \circ'_{\bar{C}} f$. Therefore, the semi-group $\langle C, \circ'_{\bar{C}} \rangle$ is an extension of quasi-algebra $\langle C, \circ_C \rangle$ and this means that there exists an extension of the quasi-algebra $\langle A, \circ \rangle$ to a semi-group.

REFERENCES

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