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# *THE PROJECTIVE LIMIT OF THE $H^p$ SPACES*

BY

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**1. Introduction.** If $f(z) = z + a_2 z^2 + \ldots$ is analytic and univalent in the open unit disk, then $\log\big(f(z)/z\big)$ belongs to $H^p$ for all $0 < p < \infty$, and hence to $\bigcap_{p<\infty} H^p$. As sets, $\bigcap_{p<\infty} H^p$ and $\varprojlim H^p$, the projective limit of the $H^p$ spaces ($1 \leqslant p < \infty$), can be identified. These observations suggest that there are interesting connections between various classes of conformal mappings and the projective limit. In this paper* we explore some of these connections and we also develop various properties of the projective limit.

First we observe that $\varprojlim H^p$ is a Fréchet space with polynomials dense, but it is not normable. It is shown (Theorem 2.1) that $\varprojlim H^p$ and the projective limits of sets of boundary functions and $L^p$ functions are related in a way similar to the situation in each individual $H^p$ space. In the third section of this paper we consider sets of functions $\log\big(f(z)/z\big)$ for $f$ running over the set of normalized univalent functions and for $f$ close-to-convex, or starlike, or spirallike. We examine the closure, convexity, extreme points and isolated points of these sets $\{\log\big(f(z)/z\big)\}$ as subsets of the projective limit $\varprojlim H^p$.

In Section 4 we consider operators on $\varprojlim H^p$. An example is given which shows that a continuous operator on $\varprojlim H^p$ need not extend to a continuous operator on $H^q$ and need not be continuous on $\bigcap H^p$ with the topology induced by $H^q$ (some $q \in (1, \infty)$). We also list a few results about composition operators on $\varprojlim H^p$. In Section 5 we give a direct proof that the dual of $\varprojlim H^p$ is the inductive limit of the duals $(H^p)^*$, $1 \leqslant p < \infty$. An example shows that the dual $(\varprojlim H^p)^*$ is a proper subset of $H^1$. We also show that $\varprojlim H^p$ is reflexive and that $(\varprojlim H^p)^*$ is complete.

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* This work is based on a portion of the second author's Ph. D. dissertation [5].

**2. The space $\varprojlim H^p$.** In order to define the projective limit of the $H^p$ spaces we introduce the Cartesian product $\prod H^p$ $(1 \leqslant p < \infty)$ and denote its elements by $\tilde{f} = (f_p)$, $f_p \in H^p$ for each $p \in [1, \infty)$. For each $q \in [1, \infty)$, let $\pi_q$: $\prod H^p \to H^q$ be the canonical projection $\pi_q((f_p)) = f_q$ and, for each $p$ with $1 \leqslant q \leqslant p < \infty$, let $\iota_{pq}$: $H^p \to H^q$ be the inclusion map. The topology on $\prod H^p$ is generated by sets of the form $\prod W_p$, where each $W_p$ is open in $H^p$ and $\pi_p(W_p) = H^p$ for all but a finite number of factors, i.e., the topology of "pointwise convergence", since a net converges in the product if and only if its projection converges in each factor.

Definition 2.1. The *projective limit* of the $H^p$ spaces for $1 \leqslant p < \infty$, written $\varprojlim H^p$, is the set of all elements $\tilde{f} = (f_r)$ in $\prod H^p$ such that, whenever $1 \leqslant q \leqslant p < \infty$,

$$\pi_q(\tilde{f}) = \iota_{pq} \circ \pi_p(\tilde{f}).$$

The topology is the inherited topology.

Clearly, $\tilde{f} = (f_r)$ is in $\varprojlim H^p$ if and only if $f_p = f_q$ for all $p, q \in [1, \infty)$, and $\varprojlim H^p$ is a subspace of $\prod H^p$. Note that $\varprojlim H^p$ is topologically isomorphic to $\bigcap H^p$, where the intersection has the topology generated by all sets of the form $V \cap (\bigcap H^p)$, and $V$ is open in some $H^q$. Furthermore, the restriction of the projection $\pi_p$ to $\varprojlim H^p$ (hence to $\bigcap H^p$) is continuous, and for each $q$ we can consider the set $\bigcap H^p$ as a subspace of $H^q$ with the inherited topology. For additional information about projective and inductive limits we refer the reader to Schaefer [9], Kelley et al. [6], and Dugundji [3].

The integers are cofinal in the real numbers and, therefore,

$$\varprojlim H^p = \varprojlim H^n,$$

where $\varprojlim H^n$ is considered as a subspace of the countable product $\prod_{n=1}^{\infty} H^n$. The countable product $\prod H^n$ is a complete metric space with metric

$$\|(f_n) - (g_n)\| = \sum_{n=1}^{\infty} 2^{-n} \min\{1, \|f_n - g_n\|_n\}. \tag{2.1}$$

It follows easily that $\varprojlim H^p$ is a Fréchet space. By a standard argument one can show that the projective limit is not normable. For norm (2.1) and $f \in H^\infty$ we have $\|(f)\| \leqslant \|f\|_\infty$, and hence the inclusion map $H^\infty \to \varprojlim H^p$ is continuous.

The relationship between boundary values and the polynomials in $L^p = L^p[0, 2\pi]$ carries over to $\varprojlim H^p$. We let $\mathscr{H}^p$ denote the set of boundary functions $f(e^{it})$ for $f \in H^p$, and let $Q^p$ denote the set of polynomials as a subset of $L^p$.

THEOREM 2.1. *The space* $\varprojlim \mathscr{H}^p$ *is the* $\varprojlim L^p$ *closure of the set of polynomials* $\varprojlim Q^p$.

Proof. Since $\mathscr{H}^p = \overline{Q^p}$ (the $L^p$ closure), we have

$$\prod \mathscr{H}^p = \prod \overline{Q^p} \quad \text{and} \quad \varprojlim \mathscr{H}^p = \varprojlim \overline{Q^p}.$$

Furthermore, $\prod \overline{Q^p} = \mathrm{cl}\left(\prod Q^p\right)$, where cl denotes the closure in $\prod L^p$. Since $\overline{Q^p}$ is Hausdorff, $\varprojlim \overline{Q^p}$ is closed in $\prod \overline{Q^p}$, and hence in $\prod L^p$. It follows that

$$\mathrm{cl}(\varprojlim Q^p) \subset \varprojlim \overline{Q^p} = \varprojlim \mathscr{H}^p.$$

To show equality, we let $\pi_j^{-1}(W)$ be a basic open set in $\varprojlim \mathscr{H}^p$, where $W$ is an open set in $\mathscr{H}^j$. Then there is a polynomial

$$q \in \Pi_j^{-1}(W) \cap \varprojlim Q^p,$$

which shows that $\varprojlim Q^p$ is dense in $\varprojlim \mathscr{H}^p$, and hence

$$\mathrm{cl}(\varprojlim Q^p) = \varprojlim \mathscr{H}^p.$$

Since $\mathscr{H}^p$ $(1 \leqslant p \leqslant \infty)$ is the class of $L^p$ functions whose negative Fourier coefficients vanish, the following result is immediate:

THEOREM 2.2. *The space* $\varprojlim \mathscr{H}^p$ *is the class of functions in* $\varprojlim L^p$ *whose negative Fourier coefficients vanish.*

Various other results of the $H^p$ theory relating to notions of invertibility, zeros, outer functions, etc. can be properly interpreted and carried over to the projective limit setting [5]. Finally, we mention that pointwise multiplication in $\varprojlim H^p$ is defined (componentwise) and continuous.

**3. Schlicht functions and $\varprojlim H^p$.** We let $\mathscr{S}$ denote the class of functions

$$f(z) = z + a_2 z^2 + \ldots \tag{3.1}$$

that are analytic and schlicht (one-to-one) in the open unit disk $D = \{z\colon |z| < 1\}$. The normalization $f'(0) = 1$ in (3.1) is merely for convenience in the subsequent discussion and entails no essential loss of generality. The connection between $\varprojlim H^p$ and $\mathscr{S}$ is the result of the assertion that if $f \in \mathscr{S}$, then $\log\left(f(z)/z\right)$ belongs to $\bigcap H^p$ $(0 < p < \infty)$ [1]. We always mean the branch of the log that vanishes at $z = 0$.

THEOREM 3.1. *The set* $L = \left\{\left(\log(f/z)\right)\colon f \in \mathscr{S}\right\}$ *is a closed subset of* $\varprojlim H^p$.

Proof. Let $f \in \mathscr{S}$ and let $\tilde{F}(z) = \left(\log(f/z)\right)$ be the corresponding element of $\varprojlim H^p$. For each $p \in [1, \infty)$ the projection map $\pi_p(\tilde{F}) = \log(f/z)$ is continuous into $H^p$, and convergence in the $H^p$ norm implies uniform convergence on compacta. Thus, if $\tilde{F}_n = (\log(f_n/z)) \in L$ $(n = 1, 2, \ldots)$ and $\tilde{F}_n$ converges to $\tilde{F}$ in $\varprojlim H^p$, then there is a function $f \in \mathscr{S}$ such that $f_n$ converges to $f$ uniformly on compacta in $D$ (clearly, $f$ must be the same in each of the $H^p$ factors, $1 \leqslant p < \infty$) and $\tilde{F} = (\log(f/z)) \in L$.

Remark. In a similar way one can show that the set $\{(\log(f/z)) : f \in \mathscr{S}, f(D) \text{ convex}\}$ is closed in $\varprojlim H^p$. Similarly, one can replace "convex" by various other familiar subclasses of $\mathscr{S}$, e.g., "close-to-convex", "starlike", "spirallike". Our next result shows that the sets associated with classes of spirallike functions are convexa in the function space as well as closed.

We let $\mathrm{Sp}(\gamma)$ $(-\pi/2 < \gamma < \pi/2)$ denote the set of functions $f(z) = z + a_2 z^2 + \ldots$ that are analytic and satisfy $\mathrm{Re}\{e^{i\gamma} z f'(z)/f(z)\} > 0$ for $z \in D$. This is the so-called class of *$\gamma$-spirallike functions* (see [7], p. 171). Each $\mathrm{Sp}(\gamma)$ is a subset of $\mathscr{S}$, and $\mathrm{Sp}(0)$ is the class of normalized starlike functions. By means of the defining condition and the Herglotz integral representation one can show that $f(z) = z + a_2 z^2 + \ldots$ belongs to $\mathrm{Sp}(\gamma)$ if and only if there is a probability measure $dm$ on $[0, 2\pi]$ such that

$$\log \frac{f(z)}{z} = -2e^{-i\gamma} \cos\gamma \int_0^{2\pi} \log(1 - z e^{-it})\, dm(t) \qquad \text{for } z \in D \tag{3.2}$$

(cf. [2], p. 327). With representation (3.2) it is easy to see that the set $\{(\log(f/z)) : f \in \mathrm{Sp}(\gamma)\}$ inherits the convexity of the set of probability measures on $[0, 2\pi]$. Combination of this result with the observation that $\mathrm{Sp}(\gamma)$ is closed with respect to local uniform convergence yields

THEOREM 3.2. *For each $\gamma \in [-\pi/2, \pi/2]$, the set $\{(\log(f/z)) : f \in \mathrm{Sp}(\gamma)\}$ is a closed convex subset of $\varprojlim H^p$.*

By (3.2) and a slight modification of an argument given in [2], p. 331--332, we can establish the next result:

THEOREM 3.3. *For fixed $\gamma$, the point $(\log(f/z))$ is an extreme point of the set $\{(\log(f/z)) : f \in \mathrm{Sp}(\gamma)\}$ in $\varprojlim H^p$ if and only if the support of the associated measure $dm$ in (3.2) consists of a single point.*

We close this section with an observation about isolated points of the set $L$.

THEOREM 3.4. *The set $L = \{(\log(f/z)) : f \in \mathscr{S}\}$ has no isolated points in $\varprojlim H^p$.*

Proof. If $f(z) \in \mathscr{S}$, then $f_r(z) = f(rz)/r$ $(0 < r < 1)$ also belongs to $\mathscr{S}$. If we let $F(z) = \log(f(z)/z)$, then $F(z) \in H^p$ for each $p \in [1, \infty)$ and $F(rz) = \log(f_r(z)/z)$ converges to $F(z)$ in the $H^p$ topology. Hence $\tilde{F}(rz) = (F(rz)) \in L$ and converges to $\tilde{F}(z) = (\log(f(z)/z))$ in $\varprojlim H^p$.

**4. Operators on $\varprojlim H^p$.** It is of interest to investigate the bounded linear operators on $\varprojlim H^p$. We list a few results concerning a class of operators on $\varprojlim H^p$. These operators arise naturally in considering the $H^p$ spaces and their properties have been studied by Schwartz in [10]. First we observe that if $A$ is a continuous operator on an $H^q$ space $(1 \leqslant q < \infty)$ and if $A$ maps $\bigcap H^p$ into $\bigcap H^p$, then an elementary application of the closed graph theorem implies that $A$ is continuous on $\varprojlim H^p$. The following example shows that a continuous operator on $\varprojlim H^p$ need not extend to a continuous operator on $H^q$, and need not be continuous on $\bigcap H^p$ with the induced $H^q$ topology.

Fix $q$ with $1 \leqslant q < p < \infty$ and define a function $f$ in $H^q \setminus H^p$ by

$$f(z) = (1-z)^{-1/q}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-(1+\varepsilon)/q}, \quad \text{where } \varepsilon > 0.$$

Define a multiplication operator $A$ on $\varprojlim H^p$ by

$$A(h) \equiv \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{(1+\varepsilon)/q} h(z) \equiv g(z)h(z).$$

$A$ is a linear operator and is continuous on $\varprojlim H^p$. If $A$ were continuous on the space $\bigcap H^p$ with the induced $H^q$ topology, we could extend $A$ to a continuous linear operator $\hat{A}$ on $H^q$. Choose a sequence of polynomials $\{p_n\}$ converging in $H^q$ to $f$. A standard argument shows that $\hat{A}(p_n)$ tends pointwise to the function $f \cdot g$. Hence, $\hat{A}(f) = fg$, but this is absurd, since $fg$ is not in $H^q$.

We now list a few theorems on composition operators. If $\varphi$ is an analytic mapping of the unit disk into the unit disk, then $C_\varphi(f) = f \circ \varphi$ is a bounded linear operator on $H^p$ $(1 \leqslant p < \infty)$ and a sharp norm estimate for $C_\varphi$ is given by the inequality

$$\|C_\varphi\| \leqslant \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/p}.$$

Our earlier comments show that each such composition operator is a continuous linear operator on $\varprojlim H^p$. A similar number can be obtained for $C_\varphi$ acting on $\varprojlim H^p$, but since the metric on $\varprojlim H^p$ is not homogeneous of any degree, we do not denote it by $\|C_\varphi\|$. The following statement contains this estimate:

If $f \in \varprojlim H^p$ and $\varphi\colon D \to D$, then

$$\|C_\varphi(f)\| \leqslant \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\|f\|.$$

Rather than catalogue the list of theorems which hold for composition operators on $\varprojlim H^p$, we list only a sample.

THEOREM 4.1. *If $\{C_{\varphi_n}\}$ converges uniformly to $C_\varphi$ on $\varprojlim H^p$ and if each $C_{\varphi_n}$ is compact, then $C_\varphi$ is compact.*

THEOREM 4.2. *The operator $C_\varphi$ is compact on $\varprojlim H^p$ if and only if, for every sequence $\{f_n\}$ such that $f_n \to f$ subuniformly and $\{f_n\}$ is (topologically) bounded in $\varprojlim H^p$, we have $C_\varphi(f_n) \to C_\varphi(f)$ in $\varprojlim H^p$.*

**5. The dual of $\varprojlim H^p$.** For a topological vector space $X$, the dual of $X$, consisting of all continuous linear functionals from $X$ to $C$, will be denoted by $X^*$. For a projective limit $\varprojlim X^a$, the dual $(\varprojlim X^a)^*$ is known if $(\varprojlim X^a)^*$ and $(X^a)^*$ are assumed to have certain topologies (see Schaefer [9], p. 139-140). Since for $\varprojlim H^p$ we can give a more direct proof using the original topologies, and since we can prove several other results using the machinery of the direct proof, we will use basic concepts to investigate the dual of $\varprojlim H^p$. We, therefore, delay introducing the weak, strong, and Mackey topologies until the end of the section, when we need these concepts to show that $\varprojlim H^p$ is reflexive and that $(\varprojlim H^p)^*$ is complete.

For $1 < p < \infty$ the dual $(H^p)^*$ of $H^p$ is isomorphic to $H^q$, where $1/p + 1/q = 1$. The dual of $H^1$ can be identified with a subspace of $L^\infty$. In either case, the action is given by

$$\lambda_g(f) = \int f(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}, \quad \text{where } f \in H^p,\ g \in H^q, \text{ and } \lambda_g \in (H^p)^*$$

(see Duren [4], Chapter 7). We will write either $g$ or $\lambda_g$, depending on the context.

If $\lambda_g \in (H^p)^*$, then $\lambda_g \circ \pi_p$ is a continuous linear functional on $\varprojlim H^p$. This shows that $\bigcup_{1 \leqslant p < \infty} (H^p)^*$, which we abbreviate to $\bigcup (H^p)^*$, is contained in $(\varprojlim H^p)^*$. Since also $\bigcup (H^p)^* \subset H^1$, we would like to know whether $(\varprojlim H^p)^*$ is actually all of $H^1$ or even larger. The following example shows that $H^1$ is too large to be $(\varprojlim H^p)^*$.

Example. There is an $f$ in $\varprojlim H^p$ and a $g$ in $H^1$ such that

$$\lambda_g(f) = \int f(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}$$

is not finite. Let $f(z) = \left(-z^{-1}\log(1-z)\right)^{1+\varepsilon}$ for some $\varepsilon > 0$. Since $z/(1-z)$ is schlicht, $-\log(1-z)$ is in $\varprojlim H^p$, and so is $f$. Let

$$g(z) = (1-z)^{-1}\left(-z^{-1}\log(1-z)\right)^{-(1+\varepsilon)}.$$

By Privalov [8], p. 59, $g \in H^1$, but $g \notin H^p$ for $p > 1$.

For $0<\theta<2\pi$ we can write $1-e^{i\theta} = 2\sin(\theta/2)e^{i(\theta-\pi)/2}$. Using this, we have

$$\log(1-e^{i\theta}) = \log\left(2\sin\frac{\theta}{2}\right)+i\frac{\theta-\pi}{2} \equiv \mu+i\nu,$$

and we define $\psi$ by

$$\psi(\theta) \equiv \arg(\mu+i\nu) = \arctan\left(\frac{\nu}{\mu}\right).$$

With this notation, the integral notation for $\lambda_g$ is

$$\begin{aligned}\lambda_g(f) &= \int (1-e^{-i\theta})^{-1}\exp\{2(1+\varepsilon)(\pi-\theta+\psi(\theta))\}\frac{d\theta}{2\pi}\\ &=\frac{1}{2}\int \exp\{2(1+\varepsilon)(\pi-\theta+\psi(\theta))\}\frac{d\theta}{2\pi}-\\ &-\frac{i}{2}\int\frac{\cos(\theta/2)}{\sin(\theta/2)}\cos\{2(1+\varepsilon)(\pi-\theta+\psi(\theta))\}\frac{d\theta}{2\pi}+\\ &+\frac{1}{2}\int\frac{\cos(\theta/2)}{\sin(\theta/2)}\sin\{2(1+\varepsilon)(\pi-\theta+\psi(\theta))\}\frac{d\theta}{2\pi}\\ &\equiv\frac{1}{2}I_1+\left(\frac{-i}{2}\right)I_2+\frac{1}{2}I_3.\end{aligned}$$

The integral $I_1$ is bounded and, by elementary symmetry properties, one can show that $I_2=0$. The same symmetry properties allow us to rewrite $2^{-1}I_3$ in the form

$$2^{-1}I_3 = \int_0^{\pi}\frac{\cos(\theta/2)}{\sin(\theta/2)}\sin\{2(1+\varepsilon)(\pi-\theta+\psi(\theta))\}\frac{d\theta}{2\pi}.$$

We have

$$\pi+\psi(\theta) = \arctan\frac{(\pi-\theta)/2}{-\log(2\sin(\theta/2))}$$

and for $\theta>0$ near zero the approximation

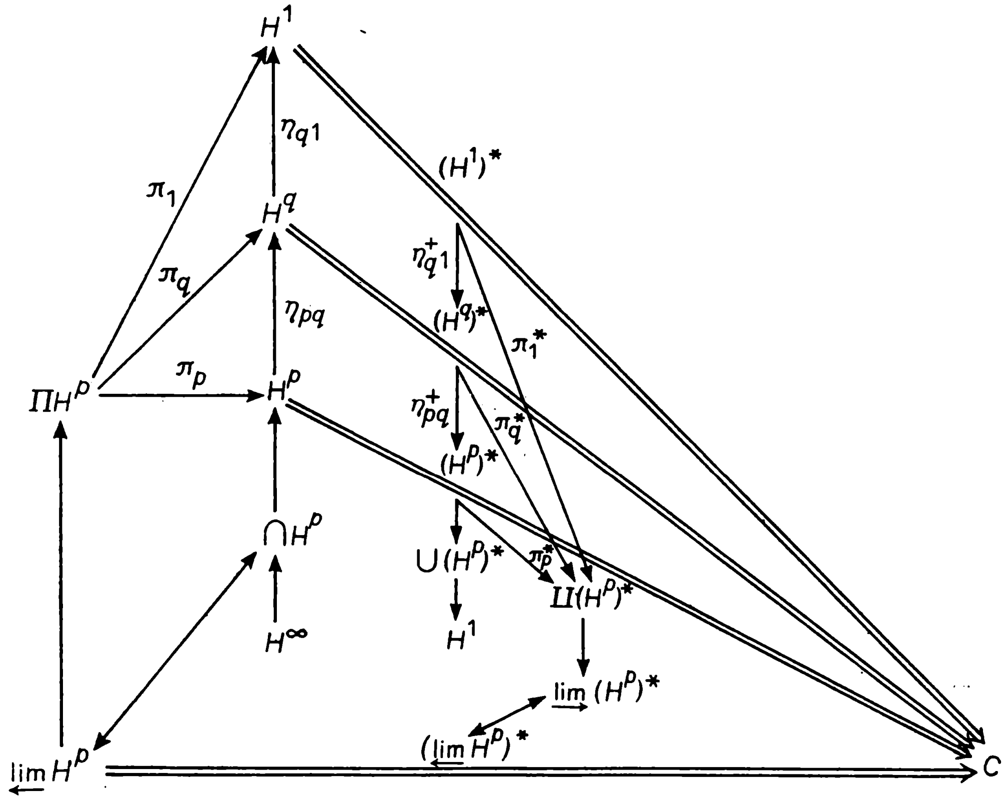
$$\pi+\psi(\theta) \approx \frac{(\pi-\theta)/2}{-\log(2\sin(\theta/2))}$$

is valid. In particular, for these $\theta$ the integrand of $I_3$ is approximately

$$2(1+\varepsilon)\frac{\cos(\theta/2)}{\sin(\theta/2)}\left\{\frac{(\pi-\theta)/2}{-\log\theta}-\theta\right\}\simeq\left\{\frac{\cos(\theta/2)(\pi-\theta)}{-\theta\log\theta}-2\cos\frac{\theta}{2}\right\}\cdot 2(1+\varepsilon),$$

where we have used the approximation $\sin x \approx x$ for $x$ near zero. This function behaves as the function $(\theta \log \theta)^{-1}$ on any small interval $[0, t]$, and so $I_3$ is a divergent integral.

In the remainder of this section we frequently refer to definitions and some theorems which appear in the standard texts of Kelley et al. [6] and Schaefer [9]. In order to keep the exposition clear and concise we will include only those results and definitions imperative for our development and give specific references to all others. A diagram is included to indicate the various mappings and inclusions (Fig. 1).




Definition 5.1. The coproduct or direct sum of the spaces $(H^p)^*$, $1 \leqslant p < \infty$, is denoted by $\coprod (H^p)^*$ and is the subspace of $\prod (H^p)^*$ consisting of $(g_p)$, $g_p \in (H^p)^*$, such that $g_p = 0$ for all but a finite number of $p$. The injection

$$\pi_p^*\colon (H^p)^* \to \coprod (H^p)^*$$

is defined as follows: for $g \in (H^p)^*$, $\pi_p^*(g)$ is the element $(g_q)$ of $\coprod (H^p)^*$ for which $g_q = 0$ if $q \neq p$, and $g_p = g$. The topology for $\coprod (H^p)^*$ is generated by all subsets $W$ of $\coprod (H^p)^*$ such that, for each $p$, $\pi_p^{*-1}(W)$ is a neighborhood of 0 in $(H^p)^*$.

By this definition, $\coprod (H^p)^*$ becomes a locally convex Hausdorff space.

Definition 5.2. For any $p$ and $q$ with $1 \leqslant q \leqslant p < \infty$, let $\eta^+_{pq}$: $(H^q)^* \to (H^p)^*$ be the inclusion map.

Each $\eta_{pq}^{+}$ is, of course, continuous and, for $1 \leqslant r \leqslant q \leqslant p < \infty$,

$$\eta_{pr}^{+} = \eta_{pq}^{+} \circ \eta_{qr}^{+}$$

holds, since each map is inclusion. Thus $\{(H^p)^*\colon \eta_{pq}^{+}\}$ is an inductive spectrum.

Definition 5.3. The *inductive limit* of the spaces $(H^p)^*$ with $1 \leqslant p < \infty$, written $\varinjlim (H^p)^*$, is the quotient $\coprod (H^p)^*/N$, where $N$ is the subspace of $\coprod (H^p)^*$ generated by the ranges of $\pi_q^* - \pi_p^* \circ \eta_{pq}^{+}$ for all $1 \leqslant q \leqslant p < \infty$. The topology of $\varinjlim (H^p)^*$ is the quotient topology.

Since the integers are cofinal in the real numbers, we have

$$\varinjlim (H^p)^* = \varinjlim (H^n)^*,$$

where $\varinjlim (H^n)^*$ is the quotient of $\coprod_{n=1}^{\infty} (H^n)^*$. For the same reason we may ignore $(H^1)^*$, and do so since it is not another $H^p$ space.

From Kelley et al. [6], p. 120, we see that

$$(\varprojlim H^p)^* = \Big(\prod H^n\Big)^* \Big/ (\varprojlim H^p)^{\perp} \quad \text{and} \quad \Big(\prod H^n\Big)^* = \coprod (H^n)^*,$$

where the second correspondence is defined as follows: if $\lambda \in \left(\prod H^n\right)^*$, then there is a $(\lambda_n) \in \coprod (H^n)^*$ such that

$$\lambda(f_n) = \sum \lambda_n(f_n) \quad \text{for } (f_n) \in \prod H^n.$$

The sum is finite, since all but finitely many $\lambda_n$ are 0. The topology of $(\varprojlim H^p)^*$ is the quotient topology.

The proof of the first isomorphism in the next theorem is due to B. S. Rajput.

THEOREM 5.1. *The space* $(\varprojlim H^p)^*$ *is topologically isomorphic to* $\varinjlim (H^p)^*$ *and to* $\bigcup (H^p)^*$, *where the topology on the union is given as follows:* $V$ *is open in* $\bigcup (H^p)^*$ *if and only if* $V \cap (H^p)^*$ *is open in* $(H^p)^*$ *for each* $p$, $1 \leqslant p < \infty$.

Proof. We first show the topological isomorphism of $\varinjlim (H^p)^*$ and $\bigcup (H^p)^*$. Let $\Phi\colon \coprod (H^p)^* \to \bigcup (H^p)^*$ be defined by $\Phi(g_p) = \sum g_p$. The sum is, of course, finite. We will show that $\ker \Phi = N$, where $N$ is defined in Definition 5.3.

Let $(g_q) \in \ker \Phi$. Then $g_q$ is non-zero only for $q = q_1, \ldots, q_m$. Let $p = \max\{q_1, \ldots, q_m\}$. The element

$$\sum (\pi_{q_i}^* g_{q_i} - \pi_p^* \eta_{pq_i}^{+} g_{q_i})$$

where $\lambda_g$ corresponds to $(g_p)$ in the coproduct. The sum is finite. We will show that, as sets, $N^{\perp} = \varprojlim H^p$.

Let $f \in \varprojlim H^p$. If $g$ is a generating element of $N$, then

$$(*) \qquad g = \pi_s^* g_s - \pi_q^* \eta_{qs}^+ g_s \quad \text{for some } 1 \leqslant s \leqslant q < \infty \text{ and } g_s \in (H^s)^*.$$

For $\varphi$ in $\left(\coprod (H^p)^*\right)^*$ corresponding to $f \in \varprojlim H^p \subset \prod H^p$, we have

$$\varphi(\lambda_g) = \sum \int f_p(e^{i\theta})\overline{g_p(e^{i\theta})}\frac{d\theta}{2\pi}$$
$$= \int f(e^{i\theta})\overline{g_s(e^{i\theta})}\frac{d\theta}{2\pi} - \int f(e^{i\theta})\overline{g_s(e^{i\theta})}\frac{d\theta}{2\pi} = 0,$$

hence $f \in N^{\perp}$.

Let $(f_p) \in N^{\perp}$, and let $\varphi \in \left(\coprod (H^p)^*\right)^*$ correspond to $(f_p)$. Let $g$ defined by $(*)$ be in $N$. Since $(f_p) \in N^{\perp}$, we have

$$\varphi(\lambda_g) = \sum \int f_p(e^{i\theta})\overline{g_p(e^{i\theta})}\frac{d\theta}{2\pi} = \int f_s(e^{i\theta})\overline{g_s(e^{i\theta})}\frac{d\theta}{2\pi} - \int f_q(e^{i\theta})\overline{g_q(e^{i\theta})}\frac{d\theta}{2\pi}$$
$$= \int \left(f_s(e^{i\theta}) - f_q(e^{i\theta})\right)\overline{g_s(e^{i\theta})}\frac{d\theta}{2\pi} = 0.$$

Since this holds for every $g_s \in (H^s)^*$, we must have $f_s = f_q$. Since we can do this for any $s$ and $q$, we must have $f_p = f_s$ for all $p$. Thus

$$f = (f_p) = (f_s) \in \varprojlim H^p,$$

and the proof of Theorem 5.3 is complete.

This suggests that $\varprojlim H^p$ may be reflexive. To investigate this and other questions we need the definition of weak, Mackey and strong topology on a locally convex space $E$. We also need the definitions of the terms "barrelled", "semi-reflexive", "evaluable" and "reflexive". These are readily available in the two standard texts referred to above. We quote, for reference purposes, a collection of results basic for our final theorems.

THEOREM 5.4. (a) *Every Fréchet space is barrelled* (Schaefer [9], p. 60).

(b) *The inductive limit of a family of barrelled spaces is barrelled* (Schaefer [9], p. 60).

(c) *If the original topology on $E$ is barrelled, then the original topology and $m(E, E^*)$ coincide* (Schaefer [9], p. 132).

(d) *Every barrelled space is evaluable* (Kelley et al. [6], p. 193).

(e) *The projective limit of a family of semi-reflexive locally convex spaces is semi-reflexive* (Schaefer [9], p. 146).

(f) *If $E$ is locally convex and semi-reflexive, then $m(E^*, E) = s(E^*, E)$* (Kelley et al. [6], p. 190).

(g) *A locally convex Hausdorff space is reflexive if and only if it is semi-reflexive and evaluable* (Kelley et al. [6], p. 194).

(h) *If $E$ is locally convex and metrizable, then $E^*$ is strongly complete.*

THEOREM 5.5. *The space* $\varprojlim H^p$ *is reflexive.*

Proof. From Theorem 5.4 (a) and (d) it follows that $\varprojlim H^p$ is evaluable. By (e), it is semi-reflexive. By (g), $\varprojlim H^p$ is reflexive.

THEOREM 5.6. *On the space* $(\varinjlim H^p)^*$, *the topologies*

$$m\left(\varinjlim (H^p)^*, \varprojlim H^p\right) \quad \text{and} \quad s\left(\varinjlim (H^p)^*, \varprojlim H^p\right)$$

*coincide with the original topology.*

Proof. By Theorem 5.4 (b) and (c), the original topology coincides with $m\left(\varinjlim (H^p)^*, \left(\varinjlim (H^p)^*\right)^*\right)$, where $\left(\varinjlim (H^p)^*\right)^*$ is the dual with respect to the original topology on $\varinjlim (H^p)^*$. By Theorem 5.3 the original topology coincides with $m\left(\varinjlim (H^p)^*, \varprojlim H^p\right)$, which coincides with $s\left(\varinjlim (H^p)^*, \varprojlim H^p\right)$ by Theorem 5.4 (f).

THEOREM 5.7. *The space* $\varinjlim (H^p)^*$ *is not metrizable, every bounded subset is nowhere dense, and* $\varinjlim (H^p)^*$ *is of first category in itself. Further,* $\varprojlim (H^p)^*$ *is neither first nor second countable.*

This follows from Theorem 5.6 and from Kelley et al. [6], p. 213.

THEOREM 5.8. *The space* $\varinjlim (H^p)^*$ *is complete with its original topology.*

This follows from Theorem 5.4 (h) and Theorem 5.6.

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