COLLOQUIUM MATHEMATICUM

VOL. XXXV 1976 FASC. 1

ON A CERTAIN L-IDEAL OF THE MEASURE ALGEBRA

BY

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0. Let G be an infinite compact abelian group, and $\hat{G}$ the dual group of G. Let $M(G)$ be the Banach algebra of all bounded regular Borel measures on G with total variation norm and convolution as multiplication. In a paper by Hartman and Ryll-Nardzewski [5] the following question is asked:

Does there exist a continuous measure μ on the circle group such that $\{n \in Z;\ |\hat{\mu}(n)| > a\}$ is not a Sidon set for some positive numbers a?

The existence of such a measure is showed in [4], [10] and [11]. We denote here by $\mathscr{S}(G)$ the set of all $\mu \in M(G)$ such that $\{\gamma \in \hat{G};\ |\hat{\mu}(\gamma)| > a\}$ is a Sidon set for every positive number a. In this paper, we show that $\mathscr{S}(G)$ is an L-ideal and study some of its properties.

1. For $\mu, \nu \in M(G)$, we write $\mu \ll \nu$ if μ is absolutely continuous with respect to ν. A closed ideal (subalgebra) I of $M(G)$ is called an *L-ideal* (*L-subalgebra*) if $\mu \in I$ and $\nu \ll \mu$ implies $\nu \in I$. Let $M_0(G)$ be the L-ideal of $M(G)$ which consists of all elements whose Fourier-Stieltjes transform vanishes at infinity, and $M_c(G)$ the L-ideal of all continuous measures on G. For $\mu \in M(G)$ and $\gamma \in \hat{G}$, we put $d(\gamma\mu) = \gamma d\mu$. For $\mu \in M(G)$ and $a > 0$, we put

$$E_a(\mu) = \{\gamma \in \hat{G};\ |\hat{\mu}(\gamma)| > a\}.$$

LEMMA 1 (Izuchi [7]). *A closed ideal I of $M(G)$ is an L-ideal if $\gamma\mu \in I$ for every $\mu \in I$ and every $\gamma \in \hat{G}$.*

THEOREM 1. *$\mathscr{S}(G)$ is an L-ideal and $M_0(G) \subset \mathscr{S}(G)$.*

Proof. Let $\mu, \lambda \in \mathscr{S}(G)$ and $a > 0$. Then we have

$$E_a(\mu+\lambda) \subset E_{a/2}(\mu) \cup E_{a/2}(\lambda).$$

Since $E_{a/2}(\mu)$ and $E_{a/2}(\lambda)$ are Sidon sets, $E_a(\mu+\lambda)$ is a Sidon set by [2]. Thus we have $\mu+\lambda \in \mathscr{S}(G)$. It is trivial that $\mathscr{S}(G)$ is an ideal. Suppose $\mu_n \in \mathscr{S}(G)$, $\mu \in M(G)$ and $\mu_n \to \mu$ in norm. For $a > 0$, there exists an integer n_0 such that $\|\mu_{n_0} - \mu\| < a/2$. Then we have $E_a(\mu) \subset E_{a/2}(\mu_{n_0})$, so that $\mu \in \mathscr{S}(G)$. Hence $\mathscr{S}(G)$ is closed. Suppose $\mu \in \mathscr{S}(G)$. Then it is clear that $\gamma\mu \in \mathscr{S}(G)$ for every $\gamma \in \hat{G}$. By Lemma 1, $\mathscr{S}(G)$ is an L-ideal of $M(G)$.

COROLLARY 1. *We have* $\mathscr{S}(G) \subset M_c(G)$.

Proof. The Dirac measure δ_0 does not belong to $\mathscr{S}(G)$. Since $\mathscr{S}(G)$ is an L-ideal, the assertion follows.

Let G_1 be a non-compact LCA group. Suppose that $\hat{G}$ is a dense subgroup of $\hat{G}_1$, and consider $\hat{G}$ as a discrete abelian group. The dual group G of $\hat{G}$ is then compact and abelian. Let φ: $G_1 \to G$ be the natural injective map. Since $\hat{G} \circ \varphi \subset \hat{G}_1$, we can consider $\hat{G}$ as a subset of $\hat{G}_1$. For $\mu \in M(G_1)$, we put $\Phi\mu(E) = \mu(\varphi^{-1}(E))$ for every Borel set E of G. Then Φ is an isometric isomorphism from $M(G_1)$ into $M(G)$, so that $M(G_1)$ can be considered as a subset of $M(G)$. For $\gamma \in \hat{G}$, we have $(\Phi\mu)^\wedge(\gamma) = \hat{\mu}(\gamma)$. Given two subsets M_1 and M_2 of $M(G)$, we write $M_1 \perp M_2$ if every $\mu \in M_1$ is singular with respect to M_2. We will show $\mathscr{S}(G) \perp M(G_1)$.

LEMMA 2 (see, e.g., Dunkl and Ramirez [3], p. 58). *A Sidon set cannot contain arbitrarily long arithmetic progressions.*

LEMMA 3. *Let M be an L-subspace of $M(G)$. If $\mu \notin \mathscr{S}(G)$ for every positive $\mu \in M$, then $M \perp \mathscr{S}(G)$.*

This is clear by Theorem 1.

THEOREM 2. *Under the above notations, we have* $\mathscr{S}(G) \perp M(G_1)$.

Proof. For $\mu \in M(G_1)$ with $\mu > 0$, there exists $a > 0$ such that

$$E = \{\gamma \in \hat{G}_1;\ |\hat{\mu}(\gamma)| > a\} \ni 0.$$

Let $R^n \times K$ be an open closed subgroup of $\hat{G}_1$, where K is a compact group ([12], Theorem 2.4.1).

Case I. $n \neq 0$. Since E is an open set in $\hat{G}_1$, and $\hat{G}$ is dense in $\hat{G}_1$, there exist arbitrarily long arithmetic progressions in $E \cap \hat{G}$. By Lemma 2, $E \cap \hat{G}$ is not a Sidon set in $\hat{G}$. Since $E \cap \hat{G} = E_a(\mu)$, we have $\mu \notin \mathscr{S}(G)$. Since $M(G_1)$ is an L-subalgebra, we have $\mathscr{S}(G) \perp M(G_1)$ by Lemma 3.

Case II. $n = 0$. Since $\hat{G}_1$ is not discrete, there exists an infinite compact open subgroup K of $\hat{G}_1$. If $\hat{G}_1/K$ is a finite group, then $\hat{G}_1$ is a compact group. Since G_1 is a discrete group, we have $\mathscr{S}(G) \perp M(G_1)$ by Corollary 1. If $\hat{G}_1/K$ is infinite, then the annihilator $K^\perp$ of K in G_1 is an open subgroup. Then, by regularity of μ, there exists $x_0 \in G_1$ such that $\mu(K^\perp + x_0) \neq 0$. We may suppose that $x_0 = 0$, and μ is concentrated on $K^\perp$. Then, for some $a > 0$,

$$\{\gamma \in \hat{G}_1;\ |\hat{\mu}(\gamma)| > a\}$$

contains K. Since $K \cap \hat{G}$ is an infinite subgroup of $\hat{G}$, $E_a(\mu)$ is not a Sidon set. Thus we have $\mu \notin \mathscr{S}(G)$ and $\mathscr{S}(G) \perp M(G_1)$ by Lemma 3.

COROLLARY 2. *Let G_1 be a non-compact LCA group and let G be the Bohr compactification of G_1. Then we have* $\mathscr{S}(G) \perp M(G_1)$.

Let τ be a locally compact group topology on G which is strictly stronger than the initial topology of G. By the natural embedding, we can consider $M(G_\tau) \subset M(G)$. Moreover, $\hat{G}$ is dense in $\hat{G}_\tau$ (see Inoue [6]).

COROLLARY 3. *We have* $\mathscr{S}(G) \perp M(G_\tau)$.

2. By Taylor [13], there exist a compact abelian topological semigroup S and an isometric isomorphism θ on $M(G)$ into $M(S)$, the measure algebra on S, such that

(1) $\theta(M(G))$ is a weak* dense L-subalgebra of $M(S)$;

(2) the maximal ideal space of $M(G)$ can be identified with $\hat{S}$, the set of all continuous semicharacters on S, and the Gelfand transform of $\mu \in M(G)$ is given by

$$\hat{\mu}(f) = \int_S f d\theta\mu \quad \text{for } f \in \hat{S}.$$

With the above in mind, $M(G)$ and $\hat{G}$ will be considered subsets of $M(S)$ and $\hat{S}$, respectively. The closure $\bar{\hat{G}}$ of $\hat{G}$ in $\hat{S}$ is a subsemigroup of $\hat{S}$. For $f \in \hat{S}$, we put

$$M(f) = \{\mu \in M(G);\ \theta\mu \text{ is concentrated on } O(f)\},$$

where $O(f) = \{x \in S;\ |f(x)| = 1\}$.

Definition. A closed subset E of G is called a *Dirichlet set* if, for every $\varepsilon > 0$ and for every compact subset $K \subset \hat{G}$, there exists $\gamma \in \hat{G} \setminus K$ such that $|\gamma - 1| < \varepsilon$ on E.

LEMMA 4. *If E is a Dirichlet set of G, then we have $\mathscr{S}(G) \perp M(E)$, where $M(E) = \{\mu \in M(G);\ \mu$ is concentrated on $E\}$.*

Proof. Suppose the annihilator $E^\perp$ of E is an infinite subgroup. For $\mu \in M(E)$ with $\mu \geqslant 0$, there exists $a > 0$ such that $E_a(\mu) \supset E^\perp$. Since $E^\perp$ is not a Sidon set, we get $\mu \notin \mathscr{S}(G)$. Since $M(E)$ is an L-subspace, we have $\mathscr{S}(G) \perp M(E)$ by Lemma 3. Suppose E is finite. For $\mu \in M(E)$ with $\mu \geqslant 0$, there exists $a > 0$ such that $E_a(\mu)$ contains arbitrarily long arithmetic progressions (cf. the proof of Theorem 1 of Graham [4]). Then $E_a(\mu)$ is not a Sidon set by Lemma 2. Thus we have $\mu \notin \mathscr{S}(G)$ and $\mathscr{S}(G) \perp M(E)$ by Lemma 3.

THEOREM 3. *If $f \in \bar{\hat{G}} \setminus \hat{G}$, then $\mathscr{S}(G) \perp M(f)$.*

Proof. Suppose $f \in \bar{\hat{G}} \setminus \hat{G}$. Then we easily get $\bar{f} \in \bar{\hat{G}} \setminus \hat{G}$ (where $\bar{f}$ means the complex conjugate of f). Since $\bar{\hat{G}}$ is a subsemigroup, $f \cdot \bar{f} \in \bar{\hat{G}} \setminus \hat{G}$. As $M(f) = M(f \cdot \bar{f})$, we may assume $f \geqslant 0$. Since $f \in \bar{\hat{G}} \setminus \hat{G}$, there exists a net $\{\gamma_a\}$ in $\hat{G}$ such that $\gamma_a \to f$ in the weak* topology on $\bar{\hat{G}}$. For any $\mu \in M(f)$ with $\mu \geqslant 0$,

$$\hat{\mu}(\gamma_a) \to \hat{\mu}(f) = \int_G d\mu \quad \text{and} \quad \int_G (\bar{\gamma}_a - 1) d\mu \to 0.$$

On the other hand,

$$\Big(\int_G |\bar{\gamma}_a - 1|\, d\mu\Big)^2 \leqslant \|\mu\| \int_G |\bar{\gamma}_a - 1|^2 d\mu \leqslant 2\, \|\mu\| \operatorname{Re} \int_G (1-\bar{\gamma}_a)\, d\mu.$$

Thus we have

$$\int_G |\gamma_a - 1|\, d\mu \to 0 \quad \text{ if } a \to \infty.$$

By Egorov's theorem, there exists a compact subset E of G such that $\mu(E) \neq 0$ and a subsequence $\{\gamma_n\}$ of $\{\gamma_a\}$ such that $\gamma_n \to 1$ uniformly on E. Thus E is a Dirichlet set and $\mu \notin \mathscr{S}(G)$ by Lemma 4. Since $M(f)$ is an L-subalgebra, we have $\mathscr{S}(G) \perp M(f)$.

Remark 1. Brown [1] showed that there exist many idempotents in $\bar{\hat{G}}$.

Remark 2. Let τ be a locally compact group topology on G which is strictly stronger than the initial topology of G. Since $\hat{G}_\tau \subset \bar{\hat{G}}$ (see Inoue [6]), we get Corollary 3 also from Theorem 3.

Let H be the union of all maximal groups of S. We put

$$M(H) = \{\mu \in M(G);\ \theta\mu \text{ is concentrated on } H\}.$$

COROLLARY 4. *If $\mu \in M(H)$ and $\mu \in \mathscr{S}(G)$, then $\mu \in M_0(G)$.*

Proof. Suppose $\mu \notin M_0(G)$. There exists $f \in \bar{\hat{G}} \setminus \hat{G}$ such that $\hat{\mu}(f) \neq 0$. Since $\mu \in M(H)$, we have $\mu \operatorname{non} \perp M(f)$. By Theorem 3, $\mu \notin \mathscr{S}(G)$, a contradiction.

A subset E of G is called an *H_1-set* if

$$\|\mu\| = \sup_{\gamma \in \hat{G}} |\hat{\mu}(\gamma)| \quad \text{ for every } \mu \in M(E).$$

COROLLARY 5. *If E is an H_1-set, then $\mathscr{S}(G) \perp M(E)$.*

Proof. We may assume that E is an infinite H_1-set. For $\mu \in M(E)$ with $\mu \geqslant 0$, there exists a Borel function f on E such that $|f| = 1$ on E and $|(f\mu)^\wedge(\gamma)| < \|\mu\|$ for every $\gamma \in \hat{G}$. Since E is an H_1-set, there exists a sequence $\{\gamma_n\}_{n=1}^\infty$ of distinct elements of $\hat{G}$ such that

$$|(f\mu)^\wedge(\gamma_n)| \to \|f\mu\| = \|\mu\|.$$

Let χ be a cluster point of $\{\gamma_n\}_{n=1}^\infty$ in $\hat{S}$. Then

$$\chi \in \bar{\hat{G}} \setminus \hat{G} \quad \text{ and } \quad |(f\mu)^\wedge(\chi)| = \|f\mu\| = \|\mu\|.$$

This shows that $f\mu \in M(\chi)$. By Theorem 3, $f\mu \perp \mathscr{S}(G)$ and $\mu \perp \mathscr{S}(G)$. Thus we have $\mathscr{S}(G) \perp M(E)$.

Remark 3. If E is a countable union of H_1-sets, then $M(E) \perp \mathscr{S}(G)$.

Let E be a Dirichlet set of G, and K the subgroup of G generated by E. We put

$$N(E) = \{\mu \in M(G);\ |\mu|(K+x) = 0 \text{ for every } x \in G\}$$

and

$$N(E)^{\perp} = \{\mu \in M(G);\ \mu \perp N(E)\}.$$

THEOREM 4. *If E is a Dirichlet set of G, then we have $\mathcal{S}(G) \subset N(E)$.*

Proof. Let $\mu \in N(E)^{\perp}$ and $\mu \geqslant 0$. We may assume that μ is concentrated on K. Since

$$K = \bigcup_{n=1}^{\infty} n\big(E \cup (-E)\big),$$

where $n\big(E\cup(-E)\big) = \big(E\cup(-E)\big)+\ldots+\big(E\cup(-E)\big)$ (n summands), we may, moreover, assume that μ is concentrated on $n\big(E\cup(-E)\big)$ for some integer n. On the other hand, $n\big(E\cup(-E)\big)$ is a Dirichlet set, so that $\mu \notin \mathcal{S}(G)$ by Lemma 4. Since $N(E)^{\perp}$ is an L-subalgebra, $\mathcal{S}(G) \perp N(E)^{\perp}$ by Lemma 3. Hence $\mathcal{S}(G) \subset N(E)$.

3. Here we present some other results and problems. For $\mu \in M(G)$, $\{\hat{\mu}(f);\ f \in \hat{S}\}$ is called the *spectrum of* μ. We denote by $\mathrm{Rad}\,L^1(G)$ the *radical of* $L^1(G)$, i.e., the intersection of all maximal ideals of $M(G)$ which contain $L^1(G)$.

PROPOSITION 1. *If $\mu \in \mathcal{S}(G)$ and μ has countable spectrum, then $\mu \in \mathrm{Rad}\,L^1(G)$.*

Proof. If $\mu \in \mathcal{S}(G)$ has countable spectrum, then $\mu \in M(H)$ (Izuchi [9]). By Corollary 4, we have $\mu \in M_0(G)$, so $\mu \in \mathrm{Rad}\,L^1(G)$ (Izuchi [9]).

LEMMA 5 (Izuchi [8]). *If $\mu \in M(G)$ and $E_a(\mu)$ is an infinite Sidon set for some $a > 0$, then there exists β, $0 < \beta < a$, such that $E_\beta(\mu) \setminus E_a(\mu)$ is an infinite set.*

The following proposition is clear by this lemma:

PROPOSITION 2. *If there is $a > 0$ such that $E = E_a(\mu)$ is an infinite set and $\hat{\mu}(\gamma) \to 0$ on the complement of E, then $\mu \notin \mathcal{S}(G)$.*

In Corollary 5, we show that $M(E) \perp \mathcal{S}(G)$ for every H_1-set E. But we do not know whether $M(E) \perp \mathcal{S}(G)$ for every Helson set E. Furthermore, it is an open question whether $\mathcal{S}(G) = M_0(G)$. This holds if $\mu \notin \mathcal{S}(G)$ for every $\mu > 0$ such that $\mu * \mu \perp M_0(G)$ (1).

PROPOSITION 3. *If $\mu > 0$ and $\mu * \mu \perp M_0(G)$, then there exists $a > 0$ such that $\big\{\gamma \in E;\ \{\gamma + E\} \cap E$ is infinite$\big\}$, where $E = E_a(\mu)$, is an infinite set.*

(1) Colin C. Graham has proved that $\mathcal{S}(G) = M_0(G)$ (*Non-Sidon sets in the support of a Fourier-Stieltjes transform*, to appear in this journal, vol. 36). Thus, the answers to both questions are affirmative. [Note of the Editors]

Proof. We may assume $\|\mu\| = 1$. There exists $a_1 > 0$ such that $E_{a_1}(\mu)$ is an infinite set. Let $\{\gamma_n\}_{n=1}^{\infty} \subset E_{a_1}(\mu)$ and let λ be an accumulation point of $\{\bar{\gamma}_n\mu\}_{n=1}^{\infty}$ in $M(G)$. Then $\lambda \ll \mu$ and $\lambda \neq 0$, since $|\hat{\lambda}(0)| \geqslant a_1$. In view of $\mu*\lambda \ll \mu*\mu$ and $(\mu*\lambda)\hat{}\,(0) \neq 0$, we have $\mu*\lambda \neq 0$ and $\mu*\lambda \perp M_0(G)$. Then there is a such that $a_1 > a > 0$, and $E_a(\mu*\lambda)$ is an infinite set. For $\gamma \in E_a(\mu*\lambda)$, there exists a positive integer n_0 such that

$$|\hat{\mu}(\gamma)(\bar{\gamma}_n\mu)\hat{}\,(\gamma)| = |\hat{\mu}(\gamma)\hat{\mu}(\gamma+\gamma_n)| > a \quad \text{for every } n > n_0.$$

Since $\{\gamma_n\} \subset E_a(\mu)$ and $E_a(\mu*\lambda) \subset E_a(\mu)$, the proof is complete.

REFERENCES

[1] G. Brown, *Idempotents in the closure of the characters*, Bulletin of the London Mathematical Society 4 (1972), p. 43-46.

[2] S. W. Drury, *Sur les ensembles de Sidon*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences 271 (1970), p. 162-163.

[3] C. Dunkl and D. Ramirez, *Topics in harmonic analysis*, New York 1971.

[4] C. C. Graham, *Two remarks on the Fourier-Stieltjes transforms of continuous measures*, Colloquium Mathematicum 27 (1973), p. 297-299.

[5] S. Hartman et C. Ryll-Nardzewski, *Quelques résultats et problèmes en algèbre des mesures continues*, ibidem 22 (1971), p. 271-277.

[6] J. Inoue, *Some closed subalgebras of measure algebras and a generalization of P. J. Cohen's theorem*, Journal of the Mathematical Society of Japan 23 (1971), p. 278-294.

[7] K. Izuchi, *On a zero set of Gelfand transforms of L-ideals of measure algebras*, Science Reports of the Tokyo Kyoiku Daigaku, Section A, 11 (1972), p. 227-230.

[8] — *Sidon sets and small M_0-sets*, ibidem 12 (1974), p. 146-148.

[9] — *On measures whose spectra are countable* (to appear).

[10] R. Kaufman, *Remark on Fourier-Stieltjes transforms of continuous measures*, Colloquium Mathematicum 22 (1971), p. 279-280.

[11] D. E. Ramirez, *Remark on Fourier-Stieltjes transforms of continuous measures*, ibidem 27 (1973), p. 81-82.

[12] W. Rudin, *Fourier analysis on groups*, New York - London 1962.

[13] J. L. Taylor, *The structure of convolution measure algebras*, Transactions of the American Mathematical Society 119 (1965), p. 150-166.

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Reçu par la Rédaction le 10. 8. 1974