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SIDON SETS IN DUAL OBJECTS OF COMPACT GROUPS

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Let G be a compact group, and Σ the dual object of G, i.e., the set of equivalence classes of all continuous irreducible unitary representations of G. For each $\sigma \in \Sigma$, select a fixed member $U^{(\sigma)}$ of σ with a representation space H_σ of dimension d_σ. Let $\mathscr{B}(H_\sigma)$ denote the linear space of all bounded operators on H_σ.

The $*$-algebra $\mathsf{P}_{\sigma\in\Sigma} \mathscr{B}(H_\sigma)$ will be denoted by $l(\Sigma)$; scalar multiplication, addition, multiplication and the adjoint of an element are defined coordinatewise.

For an operator X on H_σ, let $(x_1, x_2, \ldots, x_{d_\sigma})$ be the sequence of eigenvalues of positive-definite operator $|X|$ written in any order.

Let

$$\|X\|_{\varphi_p} = \varphi_p(x_1, x_2, \ldots, x_{d_\sigma}) = \Big(\sum_{i=1}^{d_\sigma} |x_i|^p\Big)^{1/p}$$

and

$$\|X\|_{\varphi_\infty} = \sup\{\|Xf\|: \|f\| = 1, f \in H_\sigma\}.$$

Let $E = (E_\sigma)$ be an element of $l(\Sigma)$. For $p \in [1, \infty)$, we define

$$\|E\|_p = \Big(\sum_{\sigma\in\Sigma} d_\sigma \|E_\sigma\|_{\varphi_p}^p\Big)^{1/p} \quad \text{and} \quad \|E\|_\infty = \sup\{\|E_\sigma\|_{\varphi_\infty}: \sigma \in \Sigma\}.$$

For $p \in [1, \infty]$, $l^p(\Sigma)$ is defined as the set of all $E \in l(\Sigma)$ for which $\|E\|_p < \infty$.

Let $\{\xi_1^{(p)}, \ldots, \xi_{d_\sigma}^{(\sigma)}\}$ be an orthonormal basis for H_σ. We define the conjugation on H_σ and D_σ by

$$D_\sigma\Big(\sum_{j=1}^{d_\sigma} a_j \xi_j^{(\sigma)}\Big) = \sum_{j=1}^{d_\sigma} \bar{a}_j \xi_j^{(\sigma)} \quad \text{and} \quad \overline{U^{(\sigma)}} = D_\sigma U^{(\sigma)} D_\sigma.$$

Let $\bar{\sigma}$ denote the equivalence class in Σ that contains the representation $\overline{U^{(\sigma)}}$. For $\mu \in M(G)$, let $\hat{\mu}(\sigma)$ be the operator on H_σ defined by

$$\langle \hat{\mu}(\sigma)\xi, \eta\rangle = \int_G \langle U_x^{(\bar{\sigma})}\xi, \eta\rangle\, d\mu(x).$$

$\hat{\mu}$ is called *Fourier-Stieltjes transform of* μ. We have $\|\hat{\mu}\|_\infty \leqslant \|\mu\|$ and $M(G)^\wedge \subset l^\infty(\Sigma)$.

Definition. Let P be a subset of Σ. We call P a *Sidon set* if, for a given $E \in l^\infty(P)$, there is a $\mu \in M(G)$ such that $\hat{\mu}(\sigma) = E_\sigma$ for $\sigma \in P$.

This implies that there is a constant $\varkappa > 0$ such that, for every $E \in l^\infty(P)$ and a suitable $\mu \in M(G)$, we have $\hat{\mu}(\sigma) = E_\sigma$ and

$$\|\mu\| \leqslant \varkappa^{-1} \|E\|_{\infty,P} = \sup\{\|E_\sigma\|_{\varphi_\infty}\colon \sigma \in P\}.$$

We then say that P is a Sidon set of *order* n.

Drury proved (see [1]) that, for a compact abelian group G, the union of any two Sidon sets in the dual group $\hat{G}$ is a Sidon set. The aim of this note is to extend this result to arbitrary compact groups by proving the following

THEOREM. *If E_1 and E_2 are Sidon sets in Σ and* $\sup\{d_\sigma\colon \sigma \in E_1\} = N$ *is finite, then the union $E_1 \cup E_2$ is a Sidon set.*

Using Drury's ideas, we first prove the following crucial lemma:

LEMMA. *If X is a finite Sidon set of order $\varkappa$, then, for each $\varepsilon \in (0, 1]$, there exists a function $f \in L^1(G)$ such that*

(a) $\hat{f}(\sigma) = I_\sigma$ *for* $\sigma \in X$,

(b) $\|\hat{f}(\sigma)\| \leqslant \varepsilon$ *for* $\sigma \in \Sigma \setminus X$,

(c) $\|f\|_1 \leqslant 4\varkappa^{-4}\varepsilon^{-1}N^4$.

Proof. Without loss of generality, we may replace the group G by $G \times \boldsymbol{T}$, Σ by $\Sigma' = \Sigma \otimes \boldsymbol{Z}$ which is the dual object of $G \times \boldsymbol{T}$, and the set X by $X' = X \times \{1\} \cup X \times \{-1\}$. Then $\bar{X}' = X'$, where, for any set $X \subset \Sigma$, the set $\bar{X}$ is defined by $(\sigma \in X) \equiv (\bar{\sigma} \in \bar{X})$.

We put

$$\boldsymbol{Z}_2 = \boldsymbol{Z}_2^{(i)} = \{1, -1\} \quad \text{and} \quad \Omega = \prod_{i=1}^{n} \boldsymbol{Z}_2^{(i)}.$$

For each $\omega \in \Omega$, there is a $\nu_\omega \in M(G)$ such that

(i) $\|\nu_\omega\| \leqslant \varkappa^{-2}N^2$,

(ii) $\hat{\nu}_\omega(\sigma_k) = d_{\sigma_k}^2 \omega_k I_{\sigma_k}$ $(k = 1, 2, \ldots, n)$,

(iii) $\sup_{\sigma \in \Sigma} \|f_\sigma\|_{A(\Omega)} \leqslant \varkappa^{-2}N^2$, where $f_\sigma(\omega) = \hat{\nu}_\omega(\sigma)$,

$$\|\hat{f}_\sigma\|_{L^1(\hat{\Omega})} = \|f_\sigma\|_{A(\Omega)} = \int_{\hat{\Omega}} \|\hat{f}_\sigma(\chi)\| d\chi$$

and

$$\hat{f}_\sigma(\chi) = \int_{\Omega} \overline{(\chi, \omega)} f_\sigma(\omega) d\omega \qquad \text{for } \chi \in \hat{\Omega}.$$

In fact, from the definition of a Sidon set, it follows that there exists a measure $\mu_\omega \in M(G)$ such that $\hat{\mu}_\omega(\sigma_k) = d_{\sigma_k}\omega_k I_{\sigma_k}$ for $\sigma_k \in X'$, and $\|\mu_\omega\| \leqslant \varkappa^{-1}N$.

We define the measure ν_ω by

$$\hat{\nu}_\omega(\sigma) \equiv \int_\Omega g_\sigma(\omega\lambda^{-1})g_\sigma(\lambda)d\lambda, \quad \text{where } g_\sigma(\lambda) = \hat{\mu}_\lambda(\sigma),\ \lambda \in \Omega,\ \sigma \in \Sigma.$$

So ν_ω has properties (i) and (ii).

To prove (iii) note that

$$\|f\|_{A(\Omega)} = \int_{\hat{\Omega}} \|\hat{f}_\sigma(\chi)\| d\chi \leqslant \int_\Omega \|f_\sigma(\omega)\| d\omega \leqslant \varkappa^{-2}N^2.$$

Now, let $\chi_\sigma(g) = \mathrm{tr}(U_g^{(\sigma)})$ be the character of the representation $U^{(\sigma)}$. We define the function R_ω on G by

$$R_\omega(g) = \prod_{k=1}^{n}\left(1+\frac{\delta\omega_k}{d_{\sigma_k}}\psi_{\sigma_k}(g)\right),$$

$$\text{where } \psi_{\sigma_k}(g) = \frac{1}{2}\left(\chi_{\sigma_k}(g)+\overline{\chi_{\sigma_k}(g)}\right),\ 0<\delta\leqslant 1.$$

Clearly, R_ω is a real-valued, non-negative function. Now consider

$$R(g) = \int_\Omega (R_\omega * \nu_\omega)(g)d\omega.$$

Next we show that $\|R\|_1 \leqslant \varkappa^{-2}N^2$. Applying Fubini's theorem and (iii), we find that

$$\|R\|_1 \leqslant \int_G\int_\Omega\int_G R_\omega(tg^{-1})d|\nu_\omega(t)|\,d\omega\,dg = \int_G\int_\Omega\left(\int_G R_\omega(g)dg\right)d|\nu_\omega(t)|\,d\omega$$

$$\leqslant \varkappa^{-2}N^2\int_G\int_\Omega R_\omega(g)dg\,d\omega = \varkappa^{-2}N^2,$$

since

$$\int_G\int_\Omega R_\omega(g)d\omega\,dg = 1.$$

Now we evaluate $\hat{R}(\sigma_k)$ for $\sigma_k \in X'$. If $\sigma_k \in X'$, then $\bar{\sigma}_k \neq \sigma_k$, and so we have

$$\hat{\chi}_{\sigma_k}(\sigma_k) = \frac{1}{d_{\sigma_k}}I_\sigma \quad \text{and} \quad \hat{\chi}_{\bar{\sigma}_k}(\sigma_k) = 0.$$

Hence we obtain

$$\hat{R}(\sigma_k) = \int_\Omega \hat{R}_\omega(\sigma_k)d_{\sigma_k}\omega_k d\omega = d_{\sigma_k}^2\int_\Omega \omega_k\left(\int_G U_g^{(\bar{\sigma}_k)}R_\omega(g)dg\right)d\omega$$

$$= d_{\sigma_k}^2\int_G U_g^{(\bar{\sigma}_k)}\left(\int_\Omega \omega_k\prod_{l=1}^{n}\left[1+\frac{\delta\omega_l}{d_{\sigma_l}}\psi_{\sigma_l}(g)\right]d\omega\right)dg$$

$$= d_{\sigma_k}^2\int_G U_g^{(\bar{\sigma}_k)}\frac{\delta}{d_{\sigma_k}}\psi_{\sigma_k}(g)dg = \frac{\delta}{2}I_{\sigma_k}.$$

Property (iii) asserts that

$$\sup_{\sigma\in\Sigma}\|f_\sigma\|_{A(\Omega)}\leqslant N^2\varkappa^{-2}.$$

Hence

$$\|\hat{R}(\sigma)\|\leqslant\int_{\hat{\Omega}}\|\hat{f}_\sigma(\chi)\|d\chi\left\|\int_{\Omega}\overline{(\chi,\omega)}\hat{R}_\omega(\sigma)d\omega\right\|$$
$$\leqslant\varkappa^{-2}N^2\left\|\int_{\Omega}\overline{(\chi,\omega)}\hat{R}_\omega(\sigma)d\omega\right\|.$$

But the character χ can be written as

$$(\chi,\omega)=\prod_k\omega_k^{\varepsilon_k},\qquad\text{where } \varepsilon_k\in\{0,1\}.$$

Hence

$$(*)\qquad\int_{\Omega}\overline{(\chi,\omega)}\hat{R}_\omega(\sigma)d\omega=\int_G U_g^{(\bar{\sigma})}\prod_{\varepsilon_k=1}\int_{\Omega_k}\left[\omega_k+\frac{\delta}{d_{\sigma_k}}\psi_{\sigma_k}(g)\right]d\omega\, dg.$$

Assuming $\sigma\in\Sigma'\setminus X'$, we have $\bar{\sigma}\neq\sigma_k$ for each k. So, if $|\{k:\varepsilon_k=1\}|\leqslant 1$, then the last expression in $(*)$ is equal to 0. If $|\{k:\varepsilon_k=1\}|\geqslant 2$, then

$$(1)\qquad\|\hat{R}(\sigma)\|\leqslant\varkappa^{-2}N^2\delta^2.$$

Conditions (a)-(c) of the Lemma will be satisfied with $\varepsilon=2\varkappa^{-2}N^2\delta$ if f is defined by its transform on Σ as follows:

$$(2)\qquad\hat{f}(\tau)=2\delta^{-1}\hat{R}(\sigma)\qquad\text{for } \sigma=(\tau,1),\ \tau\in\Sigma.$$

If a Sidon set P is infinite, then there exists a measure $\mu\in M(G)$ satisfying the conditions of the Lemma.

To prove it consider the family $\mathscr{F}$ of all finite subsets of P. So $\mathscr{F}$ is a directed set under inclusion. For $X\in\mathscr{F}$, let f_X be a function defined by (2). Then $\{f_X\}_{X\in\mathscr{F}}$ is a net in $M(G)$ and it is a $*$-weakly compact set in $M(G)$, since $\|f_X\|_1\leqslant 4\varkappa^{-4}N^4\varepsilon^{-1}$. Thus the net $\{f_X\}$ admits a subnet $\{\mu_a\}$ which converges in the $*$-weak topology of $M(G)$ to a measure $\mu\in M(G)$. Clearly,

$$\|\mu\|\leqslant 4\varkappa^{-4}N^4\varepsilon^{-1},\qquad\hat{\mu}(\sigma)=\lim_a\hat{\mu}_a(\sigma)\qquad\text{for all } \sigma\in\Sigma,$$

and this implies that

(a′) $\hat{\mu}(\sigma)=I_\sigma$ for $\sigma\in P$,

(b′) $\sup_{\sigma\in\Sigma\setminus P}\|\hat{\mu}(\sigma)\|\leqslant\varepsilon$.

Now we can obtain the Theorem very easily.

From [2], p. 416, we have the following characterization of Sidon sets:

(**) *E is a Sidon set if and only if, for each* $W \in \prod_{\sigma \in E} U(H_\sigma)$ *(where* $U(H_\sigma)$ *denotes the group of unitary operators on the Hilbert space* H_σ*), there is a measure* $\mu \in M(G)$ *such that*

$$\sup\{\|W_\sigma - \hat{\mu}(\sigma)\|:\ \sigma \in E\} < 1.$$

Suppose that E_1 and E_2 are Sidon sets, and let $E = E_1 \cup E_2$. We may suppose that E_1 and E_2 are disjoint. By the Lemma, there are measures μ_1 and μ_2 such that $\hat{\mu}_2(\sigma) = W_\sigma$ for $\sigma \in E_2$, $\hat{\mu}_1(\sigma) = W_\sigma - \hat{\mu}_2(\sigma)$ for $\sigma \in E_1$ and $\|\hat{\mu}_1(\sigma)\| \leqslant \varepsilon$ for $\sigma \in \Sigma \setminus E_1$.

To get the assertion one has only to observe that the measure $\nu = \mu_1 + \mu_2$ satisfies (**).

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