COLLOQUIUM MATHEMATICUM

VOL. LII 1987 FASC. 1

SOME SINGULAR MEASURES ON THE CIRCLE WHICH IMPROVE L^p SPACES*

BY

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1. Introduction. Convolution operators are central among the objects studied in Harmonic Analysis, and just what one can or cannot do with such operators is a leading question. Here we are concerned with a particular manifestation of the smoothing caused by convolution. Specifically, let μ be a positive Borel measure on the circle group T. If, for $p \geqslant 1$, there is a $q > p$ such that μ defines a bounded convolution operator from $L^p(T)$ to $L^q(T)$, then we shall say that μ is *L^p-improving* or that μ *improves L^p*. It is precisely this property which interests us.

Of course L^p-improving measures abound on the circle. That many absolutely continuous measures have this property is easily to see by using Young's inequality. Much more interesting is the fact that there are probability measures, singular with respect to Lebesgue measure, yet also improving $L^p(T)$ for $p > 1$. For instance, in [3], Bonami proved that certain Riesz product measures have this property, and she gave bounds on the amount of improvement. An earlier example, due to Wiener and Wintner, concocted for quite different purposes, and found in [11], vol. II, p. 146, apparently prompted E. M. Stein to pose the problem of characterizing such positive measures; see [10]. Subsequently, in [8], Oberlin proved that the Cantor–Lebesgue measure supported by the classical middle-third Cantor set is L^p-improving for each $p > 1$, and he, like Bonami, gave bounds on the amount of improvement.

Our point of departure is Oberlin's paper. Here we prove that the

* This paper is taken from the author's dissertation, which was written at Louisiana State University while the author was a student of Professor O. Carruth McGehee.

The author sincerely thanks the referee, P. Biler, for several useful suggestions that were incorporated in the final version of the paper.

Cantor–Lebesgue measures with rational ratio of dissection improve $L^p(T)$ for $p > 1$.

Before we state our results precisely and indicate the structure of the remainder of the paper, we remark that, by using the Riesz–Thorin interpolation theorem, it is easy to see that a measure improves $L^p(T)$ for *every* $p > 1$ if it improves $L^p(T)$ for *some* $p > 1$. On the other hand, no measure that is singular with respect to Lebesgue measure can improve $L^1(T)$; see [6], p. 67.

Now realize the circle group $T = \mathbf{R}/\mathbf{Z}$ as the interval $[0, 1)$, and for $1 \leqslant p < \infty$, let $L^p(T)$ be the usual Banach space of complex-valued measurable functions whose modulus is pth power integrable with respect to normalized Lebesgue measure on T. Let $\|\cdot\|_p$ denote the corresponding norm.

For $0 < \xi < 1/2$, let

$$E(\xi) = \{\sum_{j=1}^{\infty} \varepsilon_j (1-\xi)\xi^{j-1} \colon \varepsilon_j \in \{0, 1\}\}.$$

Thus $E(\xi)$ is the Cantor set constructed with constant ratio of dissection ξ. It is well known and easy to see that $E(\xi)$ is a compact set of zero Lebesgue measure and that $E(\xi)$ is the support of the Cantor–Lebesgue measure $\mu(\xi)$, a continuous measure obtained by taking the weak* limit of the sequence of discrete probability measures $\{\mu_N\}$, where

$$\mu_N = 2^{-N} \sum_{\varepsilon \in \Pi(N)} \delta_{s(\varepsilon)}$$

with $\Pi(N) = \{0, 1\}^N$, $s(\varepsilon) = \sum_{j=1}^{N} \varepsilon_j (1-\xi)\xi^{j-1}$, and δ_x denoting the unit point mass at x; see [5], pp. 13–22, or [9].

We now state our main result.

THEOREM 1.1. *Let ξ be a rational number in* $(0, 1/2)$. *Then there is a $p < 2$, dependent on $\mu(\xi)$, such that*

$$\|\mu(\xi) * f\|_2 \leqslant \|f\|_p \quad \textit{for each } f \in L^p(T).$$

We shall prove Theorem 1.1 in Section 2 by a reduction to an analogous theorem concerning probability measures on finite cyclic groups.

Let G be a cyclic group with K elements, and for $p \geqslant 1$, let $L^p(G)$ be the usual Lebesgue space on G with norm $\|\cdot\|_p$ defined in terms of the Haar measure on G that assigns mass $1/K$ to each point. Let $P(G)$ denote the set of probability measures defined on G, and provide $P(G)$ with the topology it inherits as a subset of $M(G)$, the measure algebra on G. If $\mu \in P(G)$, let $G(\mu)$ denote the subgroup of G generated by the set $\{i-j \colon i, j \in \operatorname{supp}(\mu)\}$, where $\operatorname{supp}(\mu)$ denotes the support of μ. It turns out that Theorem 1.1 follows from the next result.

THEOREM 1.2. (a) *If $\mu \in P(G)$, where G is a finite cyclic group, then there is a $p < 2$, dependent on μ, such that*

$$\|\mu * x\|_2 \leqslant \|x\|_p \tag{1}$$

for every $x \in L^p(G)$ if, and only if,

$$G(\mu) = G. \tag{2}$$

(b) *In addition, if C is a compact subset of $P(G)$ with every μ in C satisfying* (2), *then there is a $p < 2$, dependent on C, such that* (1) *is true for every $\mu \in C$ and every $x \in L^p(G)$.*

We shall prove Theorem 1.2 in Section 3. We shall conclude the paper with a section where we shall transfer the results given in Theorem 1.1 from the circle to the real line.

2. Proof of Theorem 1.1, using Theorem 1.2. This is a variant of the proof of Lemma 1 of [8].

Let $\xi = J/K$, where J and K are relatively prime positive integers satisfying $2J < K$, and for $N \geqslant 1$, let

$$G_N = \{j/K^N\colon j = 0, \dots, K^N - 1\}.$$

Then G_N is the realization of the cyclic group of K^N-th roots of 1 on the interval $[0, 1)$. For $q \geqslant 1$, let $L^q(G_N)$ denote the usual Lebesgue space formed on G_N with respect to normalized counting measure, and let $\|\cdot\|_{q,N}$ denote the norm on that space.

For each $N \geqslant 1$, the support of the approximating measure μ_N lies in the subgroup G_N, and thus an elementary limit argument suffices to prove the theorem once we know there is a $p < 2$ so that

$$\|\mu_N * f\|_{2,N} \leqslant \|f\|_{p,N}, \quad f \in L^p(G_N), \tag{3}$$

for $N \geqslant 1$.

We now sketch that limit argument. Let $C(T)$ denote the set of continuous functions on the circle, and let $\|\cdot\|_\infty$ denote the supremum norm on that space. Let $\mu = \mu(J/K)$. Evidently, it suffices to show

$$\|\mu * f\|_2 \leqslant \|f\|_p \quad \text{for } f \in C(T). \tag{4}$$

Suppose $f \in C(T)$, and for $N \geqslant 1$, let f_N be the restriction of f to G_N. Also, for $N \geqslant 1$, let

$$I(N, j) = [(j-1)/K^N,\ j/K^N) \quad \text{for } j = 1, \dots, K^N.$$

Then, since f is uniformly continuous, it follows from

$$\mu * f(x) - \mu_N * f(x) = \sum_{j=1}^{K^N} \int_{I(N,j)} [f(x-y) - f(x-(j-1)/K^N)]\, d\mu(y)$$

that

(5) $$\|\mu * f - \mu_N * f\|_\infty \to 0 \quad \text{as} \quad N \to \infty.$$

Likewise, from

$$\|\mu_N * f\|_2^2 - \|\mu_N * f_N\|_{2,N}^2 = \sum_{j=1}^{K^N} \int_{I(N,j)} \left[|\mu * f(x)|^2 - |\mu_N * f_N((j-1)/K^N)|^2\right] dx$$

and

$$\mu_N * f(x) - \mu_N * f_N\left(\frac{j-1}{K^N}\right) = \sum_{i=1}^{K^N} \left[f\left(x - \frac{i-1}{K^N}\right) - f\left(\frac{j-i}{K^N}\right)\right] \mu_N\left(\left\{\frac{i-1}{K^N}\right\}\right)$$

it is easy to see that

(6) $$|\,\|\mu_N * f\|_2 - \|\mu_N * f_N\|_{2,N}| \to 0 \quad \text{as} \quad N \to \infty.$$

Similarly,

(7) $$|\,\|f\|_p - \|f_N\|_{p,N}| \to 0 \quad \text{as} \quad N \to \infty.$$

But (5)–(7), together with (3), yield the truth of (4). Hence, we need only to prove (3).

The proof that there is a $p < 2$ so that (3) is true for each $N \geqslant 1$ is an induction argument originating in the structure of μ_N. To show this, for convenience we shall work on quotients of the integer group, $\mathbf{Z}$, which are isomorphic to the G_N's. Thus, for $N \geqslant 1$, we take $G_N = \mathbf{Z}/K^N\mathbf{Z} = \{0, \ldots, K^N - 1\}$ with the group operation being addition modulo K^N.

Now μ_N is supported on the 2^N points of G_N of the form

(8) $$K\left[\sum_{j=1}^{N-1} \varepsilon_j (K-J) J^{j-1} K^{(N-1)-j}\right] + \varepsilon_N (K-J) J^{N-1},$$

where $\varepsilon_j \in \{0, 1\}$. Therefore, if we identify μ_N with the function that assumes the value 2^{-N} on each of the points (8) and that is zero otherwise, then we may identify μ_{N-1} with the function that is 2^{1-N} at each of the points (8) when $\varepsilon_N = 0$ and that is zero otherwise. Consequently, for $N \geqslant 2$,

$$\mu_N = (1/2)(\mu_N^0 + \mu_N^{j(N)}),$$

where

$$\mu_N^0(\cdot) = \mu_{N-1}(\cdot),$$
$$\mu_N^{j(N)}(\cdot) = \mu_{N-1}(\cdot - (K-J)J^{N-1}),$$

and $j(N)$ is the element of $\{0, 1, \ldots, K-1\}$ satisfying

$$j(N) \equiv (K-J)J^{N-1} \bmod (K) \quad \text{for } N \geqslant 1.$$

Consider the sequence of probability measures $\{\nu_N\colon N \geqslant 1\}$ defined on $G = G_1$ by $\nu_N = (1/2)(\delta_0 + \delta_{j(N)})$ for $N \geqslant 1$. Evidently this sequence assumes only finitely many values. Equally important, though, is the fact that $j(N)$ is always a generator of G. Therefore, $G(\nu_N) = G$ for each $N \geqslant 1$, and hence we may apply Theorem 1.2. It follows that there is a $p_0 < 2$ so that, for $N \geqslant 1$ and $f \in L^{p_0}(G_1)$, we have

(9) $$\|\nu_N * f\|_{2,1} \leqslant \|f\|_{p_0,1}.$$

We now claim that $p = p_0$ works for each $N \geqslant 1$ in (3).

First, $\mu_1 = \nu_1$, and thus (3) is true when $N = 1$ and $p = p_0$.

Let us consider the inductive step. Suppose $N \geqslant 2$, and for the induction hypothesis, suppose (3) is true for $p = p_0$ with $N-1$ in the role of N. Now let $f \in L^p(G_N)$ with $p = p_0$, and for $j = 0, \ldots, K-1$, set

$$f_j = f \cdot \chi_{E_j}, \quad \text{where} \quad E_j = \{n \in G_N\colon\ n \equiv j \bmod (K)\}.$$

Then E_0 is isomorphic to G_{N-1}, and since $\mu_N^i * f_j$ has support in E_{i+j}, we see easily that

$$\|\mu_N * f\|_{2,N} = \left\{\frac{1}{K^N} \sum_{n=0}^{K-1} \sum_{m \in E_n} \left|\frac{1}{2}[\mu_N^0 * f_n(m) + \mu_N^{j(N)} * f_{n-j(N)}(m)]\right|^2\right\}^{1/2}$$
$$= \left\{\frac{1}{K} \sum_{n=0}^{K-1} \frac{1}{K^{N-1}} \sum_{m \in E_0} \left|\frac{1}{2}[\mu_N^0 * f_n(m+n) + \mu_N^{j(N)} * f_{n-j(N)}(m+n)]\right|^2\right\}^{1/2}.$$

In order to use the induction hypothesis, we shift everything back to E_0, the isomorph of G_{N-1}, by realizing the convolutions appearing above as convolutions of μ_N^0 with functions living on E_0. Thus, for $n = 0, \ldots, K-1$, set

$$f(n, 0)(x) = f_n(x+n)$$

and

$$f(n-j(N), j(N))(x) = f_{n-j(N)}(x+n-(K-J)J^{N-1}).$$

Then for $n = 0, \ldots, K-1$ and $m \in E_0$,

$$\mu_N^0 * f(n, 0)(m) = \mu_N^0 * f_n(m+n)$$

and

$$\mu_N^0 * f(n-j(N), j(N))(m) = \mu_N^{j(N)} * f_{n-j(N)}(m+n).$$

We are, therefore, entitled to write

$$\|\mu_N * f\|_{2,N} = \left\{\frac{1}{K} \sum_{n=0}^{K-1} \left\|\mu_{n-1} * \left[\frac{f(n, 0) + f(n-j(N), j(N))}{2}\right]\right\|_{2,N-1}^2\right\}^{1/2}.$$

By using the equation above, we invoke the induction hypothesis. From that, the triangle inequality for the norm $\|\cdot\|_{p_0,N-1}$, and the fact that

$$\|f(i, j)\|_{p,N-1}^p = K\,\|f_i\|_{p,N}^p$$

for $p = p_0$, it is easy to obtain

$$(10)\qquad \|\mu_N * f\|_{2,N} \leqslant \left\{\frac{1}{K}\sum_{n\in G_1} [(K^{1/p}/2)(\|f_n\|_{p,N}+\|f_{n-j(N)}\|_{p,N})]^2\right\}^{1/2}$$

with $p = p_0$.

Finally, by using (10), together with (9) holding for N with f replaced by $g_N(\cdot) = K^{1/p}\|f(\cdot)\|_{p,N}$ with $p = p_0$, we obtain the truth of (3) for N and $p = p_0$.

That completes the proof of (3) by induction and thus the proof of Theorem 1.1.

Remarks. (a) Evidently the proof technique of Theorem 1.1, together with Theorem 1.2 (b), reveals how a large class of L^p-improving measures may be constructed.

(b) Many more of the regularly constructed singular measures appearing in [5], pp. 13–22, may be shown to improve $L^p(T)$ by using the technique above. What we have not been able to prove that way, however, is that the Cantor–Lebesgue measure $\mu(\xi)$ improves $L^p(T)$ when ξ is irrational, for the natural approximating measures used to define $\mu(\xi)$ do not sit nicely on the circle. Whether $\mu(\xi)$ improves $L^p(T)$ when ξ is irrational appears to be an open problem (see Added in proof).

3. Proof of Theorem 1.2. To prove Theorem 1.2, we must do a certain amount of multi-variable calculus. For notation, then, we turn to [4], pp. 56–157. In addition, unless otherwise indicated, throughout this section sums will be over the cyclic group $G = \mathbf{Z}/K\mathbf{Z}$, where we shall assume $K \geqslant 2$ in order to avoid the trivial group.

Now suppose $\mu = \sum \alpha_j \delta_j$ is a probability measure on G. In Theorem 1.2 the inequality with which we must contend is

$$(11)\qquad [K^{-1}\sum_i [\sum_j \alpha_{i-j}x_j]^2]^{1/2} \leqslant [K^{-1}\sum_j x_j^p]^{1/p},$$

with x being any non-negative function on G.

The key to understanding (11) in a qualitative sense is the quadratic form obtained by considering (11) with $p = 2$. Thus, set

$$g(\mu;\, x) = \sum_i [\sum_j \alpha_{i-j}x_j]^2 - \sum_i x_i^2,$$

with x being real-valued. Then, if we identify real-valued functions on G with elements of $\mathbf{R}^K$, then the essential fact concerning g is contained in the lemma that follows.

LEMMA 3.1. *Let $G(\mu)$ be the subgroup of G generated by $D(\mu) = \operatorname{supp}(\mu) - \operatorname{supp}(\mu)$. Then $g(\mu;\, x)$ is negative semi-definite and vanishes precisely on the set*

$$Z(\mu) = \{x \in \mathbf{R}^K\colon\ x \text{ is constant on cosets of } G(\mu)\}.$$

We shall defer the proof of Lemma 3.1 to the end of the section in order first to show how Theorem 1.2 follows from it.

Proof of the necessity of $G(\mu) = G$ in Theorem 1.2 (a). We prove the contrapositive. Suppose $G(\mu) \neq G$, and let K_0 be the number of elements in $G/G(\mu)$. Then, for non-negative x in $Z(\mu)$, (11) assumes the form

$$[K_0^{-1} \sum_{j \in G/G(\mu)} x_j^2]^{1/2} \leqslant [K_0^{-1} \sum_{j \in G/G(\mu)} x_j^p]^{1/p},$$

and since $K_0 \geqslant 2$, there is a single non-negative x in $Z(\mu)$ such that this inequality fails for every $p < 2$. That completes the proof of the necessity.

To prove the sufficiency of the condition that $G(\mu) = G$ in order for (1) to hold for some $p < 2$ in Theorem 1.2, it clearly suffices to prove part (b) of the theorem. That is our next task.

Proof of Theorem 1.2 (b). First, to set the stage, let C be a compact subset of $P(G)$ with every μ in C satisfying $G(\mu) = G$, and let

$$\Delta = \{x \in \mathbf{R}^K \setminus \{0\} \colon x_j \geqslant 0 \text{ for each } j \in G\}.$$

Next, define f on $P(G) \times \Delta \times [1, 2]$ by

$$f(\mu; x; p) = [\sum_i [\sum_j \alpha_{i-j} x_j]^2]^{1/2} / [\sum_j x_j^p]^{1/p},$$

where $\mu = \sum \alpha_j \delta_j$.

Evidently inequality (11) is equivalent to

$$(12) \qquad f(\mu; x; p) \leqslant K^{1/2 - 1/p}$$

holding for $x \in \Delta$; it will be in this form that we shall treat (11). In fact, it suffices to show there is a $p_0 < 2$ such that (12) is valid for $(\mu; x; p) \in C \times \Delta \times [p_0, 2]$. To obtain that global result, we require a lemma that concerns the behavior of f near $x_0 = (1/K, \ldots, 1/K)$.

LEMMA 3.2. *There is a $p_1 < 2$, dependent on C, and there is an open neighborhood U about x_0, such that* (12) *is valid for* $(\mu; x; p) \in C \times [U \cap \sigma] \times [p_1, 2]$, *where $\sigma = \{x \in \mathbf{R}^K \colon x_j \geqslant 0$ for every j, and $\sum x_j = 1\}$ is the simplex in $\mathbf{R}^K$ spanned by the canonical basis.*

Before we prove Lemma 3.2, let us see how the global result follows from it. Here, too, Lemma 3.1 is used.

Proof of the global result, using Lemma 3.1 and Lemma 3.2. First, f is continuous, and for fixed μ and p, $f(\mu; \cdot\,; p)$ is homogeneous of degree zero, that is, purely directional. For our set of directions, then, we shall use σ.

Next, set

$$M = \max\{f(\mu; x; 2) \colon \mu \in C,\ x \in \sigma \setminus U\},$$

where U is the open neighborhood about $x_0 = (1/K, \ldots, 1/K)$ given by Lemma 3.2.

Then Lemma 3.1 implies that $M < 1$ since $f(\mu; \cdot; 2)$ assumes its maximum, 1, only on the ray $\{(x, \ldots, x): x > 0\}$.

An immediate consequence of the inequality $M < 1$ is that there is a $p_2 < 2$ such that

$$M \leqslant K^{1/2-1/p} \leqslant 1 \quad \text{for} \quad p \in [p_2, 2]. \tag{13}$$

That is just what we need in order to make the local result, Lemma 3.2, yield the global one with $p_0 = \max(p_1, p_2) < 2$.

Let us now take care of the proof of Lemma 3.2.

Proof of Lemma 3.2. We now change our perspective with respect to f. Instead of considering f as a function defined on $P(G) \times \Delta \times [1, 2]$, we now think of f as a family of functions defined on Δ and indexed in a continuous way by $P(G) \times [1, 2]$. Evidently every member of the family is C^∞ on the interior of Δ. Consequently, we shall obtain the proof of the lemma by studying the second degree Taylor expansion of the family.

First, it is easy to see that the ray $\{(x, \ldots, x): x > 0\}$ is a set of critical points for each member of the family. Thus, for $x_0 = (1/K, \ldots, 1/K)$, there is a closed ball B centered at the origin of $\mathbf{R}^K$ so that $x_0 + B$ is contained in the interior of Δ, and so that

$$f(\mu; x_0 + h; p) - K^{1/2-1/p} = q(\mu; h; p) + R_2(\mu; h; p) \tag{14}$$

for $h \in B$ and $(\mu; p) \in P(G) \times [1, 2]$, where

$$q(\mu; h; p) = (1/2) D_h^2 f(\mu; x_0; p) = (1/2)(h_1 D_1 + \ldots + h_K D_K)^2 f(\mu; x_0; p)$$

and

$$R_2(\mu; h; p) = (1/6) D_h^3 f(\mu; x_0 + \tau(\mu; h; p) \cdot h; p)$$

with $\tau(\mu; h; p) \in (0, 1)$.

A routine computation reveals that $q(\mu; h; 2) = (K/2) g(\mu; h)$. Thus, by Lemma 3.1, $q(\mu; h; 2)$ is negative semi-definite and vanishes only on the line $L = \{(x, \ldots, x): x \in \mathbf{R}\}$ whenever μ is in C.

Consequently, $q(\mu; h; 2)$ is bounded away from zero for $\mu \in C$ and $h \in \{x \in \mathbf{R}^K: \sum x_j = 0, \sum x_j^2 = 1\}$. From continuity, then, there is an $m < 0$ and a compact neighborhood of 2 in $[1, 2]$, say $[p_1, 2]$, such that

$$q(\mu; h; p) \leqslant m \tag{15}$$

for $(\mu; p) \in C \times [p_1, 2]$ and h as above.

Finally, the third order partials are bounded for

$$(\mu; x; p) \in P(G) \times [x_0 + B] \times [1, 2].$$

Thus, the limit,

$$\lim_{h \to 0} R_2(\mu; h; p)/|h|^2 = 0,$$

is uniform with respect to $\mu \in P(G)$ and $p \in [1, 2]$. Hence, there is an open ball U, centered at x_0, such that if $x_0 + h \in U$, then

(16) $$|R_2(\mu; h; p)|/|h|^2 < -m/2$$

for each $\mu \in P(G)$ and $p \in [1, 2]$. That is the last step, for when $x_0 + h \in U \cap \sigma$, we have $\sum h_j = 0$. Thus, Lemma 3.2 follows from (14), (15), and (16).

We have one task remaining in this section, and that is to prove Lemma 3.1.

Proof of Lemma 3.1. That

$$g(\mu; x) = \sum_i \left[\sum_j \alpha_{i-j} x_j\right]^2 - \sum_i x_i^2$$

is negative semi-definite follows from the fact that

$$\|\mu * x\|_2 \leqslant \|x\|_2 \quad \text{for} \quad x \in L^2(G).$$

The main problem here is seeing that g vanishes precisely on

$$Z(\mu) = \{x \in \mathbf{R}^K\colon\ x \text{ is constant on cosets of } G(\mu)\},$$

where $G(\mu)$ is the subgroup of G generated by

$$D(\mu) = \operatorname{supp}(\mu) - \operatorname{supp}(\mu).$$

We first dispose of a trivial case. When $\mu = \delta_j$ for some j, we have $D(\mu) = \{0\}$, and thus, $G(\mu) = \{0\}$. Consequently, $Z(\mu)$ is $\mathbf{R}^K$, which is consistent with the fact that $g(\mu; x) \equiv 0$ in this case. Thus, we suppose in the sequel that $\operatorname{supp}(\mu)$ contains at least two points.

Differentiating $g(\mu; x)$ with respect to x_{j_0}, we see that the set of points in $\mathbf{R}^K$ where g vanishes coincides with the solution set of the system of linear equations

$$\sum_i \left[\sum_j \alpha_{i-j} \alpha_{i-j_0} x_j\right] - x_{j_0} = 0, \quad j_0 \in G.$$

By doing a little algebra, we may re-write this homogeneous system in a much more revealing form, namely, as

(17) $$\sum_j c_{j-j_0} x_j = 0, \quad j_0 \in G,$$

where $c_0 = 1 - \sum_i \alpha_i^2$, and where $c_j = -\sum_i \alpha_{i-j}\alpha_i$ for $j \neq 0$.

Evidently, the solution set of (17) is the kernel of the linear transformation S on $\mathbf{R}^K$ given by the matrix (c_{j-i}) where $i, j = 0, \ldots, K-1$. To facilitate the study of this situation, however, it is convenient to let x assume complex values.

We now require some of the special properties of the c_j. We enumerate them:

(i) $\sum c_j = 0$;

(ii) $c_0 > 0$;

(iii) if $j \neq 0$ and $c_j \neq 0$, then $c_j < 0$; and

(iv) $\operatorname{supp}(c_j) = D(\mu) = \operatorname{supp}(\mu) - \operatorname{supp}(\mu)$.

All of these are obvious. That (ii) and (iv) fail to be true when μ is a single point mass is also clear. It is because of that, that we treated the special case at the beginning.

Using elementary properties of the matrix (c_{j-i}), found in [2], p. 242, we need only to show $\sum c_j \exp(-2\pi ijn/K)$ is non-zero for $n \in \{1, \ldots, K-1\}$ when $G(\mu) = G$, for these numbers and zero are all the eigenvalues of (c_{j-i}).

From (i)–(iii) it suffices to show that if $n \in \{1, \ldots, K-1\}$, then there is a j, dependent on n, with $c_j < 0$ and jn not divisible by K. We obtain this from a proof by contradiction.

Suppose, on the contrary, there is an $n \in \{1, \ldots, K-1\}$ such that, for each $j \in \{1, \ldots, K-1\}$ with $c_j < 0$, we have jn divisible by K. By considering the largest factor n has in common with K, it is easy to see that all of the j's with $c_j < 0$ must have a non-trivial common factor, and this factor divides K. This means, however, that $D(\mu)$ lies on a proper subgroup of G, an impossibility since $D(\mu)$ generates $G(\mu) = G$.

That completes the proof of Theorem 1.2.

Remark. The results contained in Theorem 1.1 and Theorem 1.2 have been obtained independently by W. Beckner, S. Janson, and D. Jerison; see [1] for the details.

4. Transfer. Heretofore we have worked either on the circle group or on a finite cyclic group. We now show how results like those contained in Theorem 1.1 may be transferred to the real line, $\boldsymbol{R}$. This is totally elementary, and we include it merely for the sake of completeness.

Thus, suppose μ is a positive measure on the circle, T. Then, in a natural way, we may identify μ with a unique measure ν in $M(\boldsymbol{R})$ having support in the interval $[0, 1)$ and satisfying $\hat{\nu}(j) = \hat{\mu}(j)$ for every integer j. The proposition that follows deals with the transfer mentioned.

PROPOSITION 4.1. *Let* $1 \leqslant p \leqslant q < \infty$, *and suppose there is a constant* $K > 0$ *such that*

$$\|\mu * f\|_{L^q(T)} \leqslant K \|f\|_{L^p(T)}$$

for each $f \in L^p(T)$. *Then there is a constant* $K_0 > 0$ *such that*

$$\|\nu * f\|_{L^q(\boldsymbol{R})} \leqslant K_0 \|f\|_{L^p(\boldsymbol{R})}$$

for each $f \in L^p(\boldsymbol{R})$.

An essential step in our proof of Proposition 4.1 is the reduction allowed by the following lemma, an analog of the lemma in [7].

LEMMA 4.2. *Let $\nu \in M(\boldsymbol{R})$ be a positive measure supported in the interval $[0, 1)$. If the linear operator given by convolution against ν is bounded from $L^p[0, 1)$ into $L^q(\boldsymbol{R})$ with $1 \leqslant p \leqslant q < \infty$, then the operator from $L^p(\boldsymbol{R})$ into $L^q(\boldsymbol{R})$ given by convolution against ν is also bounded.*

Although the proof of Lemma 4.2 is the same as that of the lemma in [7], we shall provide a proof here for the convenience of the reader. First, however, let us use it to prove Proposition 4.1.

Proof of Proposition 4.1. By virtue of Lemma 4.2, it suffices to show there is a $K_0 > 0$ such that

$$\|\nu * f\|_{L^q[0,2)} \leqslant K_0 \|f\|_{L^p[0,1)}$$

is true for every non-negative bounded, continuous function f defined on $[0, 1)$. For such an f, let g be the 1-periodic extension of f to $\boldsymbol{R}$. Evidently,

$$\|\nu * g\|_{L^q[1,2)} = \|\nu * g\|_{L^q[0,1)}.$$

Since a routine computation yields $\nu * f(x) \leqslant \nu * g(x)$ for $x \in [0, 2)$, we have

$$\|\nu * f\|_{L^q[0,2)} \leqslant 2^{1/q} \|\nu * g\|_{L^q[0,1)} \leqslant 2^{1/q} K \|f\|_{L^p[0,1)},$$

and the proof of Proposition 4.1 is complete.

Now let us prove Lemma 4.2.

Proof of Lemma 4.2. For each $j \in \boldsymbol{Z}$, let $E_j = [j, j+1)$, and if $f \in L^p(\boldsymbol{R})$, set

$$f_j = f \cdot \chi_{E_j}.$$

Suppose K is the norm of the operator from $L^p[0, 1)$ to $L^q(\boldsymbol{R})$ defined by convolution against ν. Then, from the translation invariance of convolution by ν, it follows that

$$\|\nu * f_j\|_{L^q(\boldsymbol{R})} \leqslant K \|f_j\|_{L^p(\boldsymbol{R})} \quad \text{for every } j.$$

Now note that the support of $\nu * f_j$ is contained in $[j, j+2)$ and that $[j, j+2)$ misses E_k unless $j = k-1$ or $j = k$. Hence, using Hölder's inequality for finite sums, we have

$$\begin{aligned}
\|\nu * f\|^q_{L^q(\boldsymbol{R})} &= \sum_k \int_{E_k} \Big|\sum_j \nu * f_j(x)\Big|^q dx \\
&\leqslant \sum_k \int_{E_k} \Big|\Big(\sum_{j=k-1}^{k} |\nu * f_j(x)|^q\Big)^{1/q} 2^{(q-1)/q}\Big|^q dx \\
&\leqslant 2^{q-1} \sum_k \int_{E_k} \sum_j |\nu * f_j(x)|^q dx = 2^{q-1} \sum_j \|\nu * f_j\|^q_{L^q(\boldsymbol{R})} \\
&\leqslant 2^{q-1} K^q \sum_j \|f_j\|^q_{L^p(\boldsymbol{R})} \leqslant 2^{q-1} K^q \|f\|^q_{L^p(\boldsymbol{R})}
\end{aligned}$$

since $p/q \leqslant 1$, and that completes the proof of the lemma.

Added in proof. The problem raised in the Remarks at the end of Section 2 has recently been solved by Michael Christ of the Princeton University. Using Littlewood–Paley theory, he has shown that $\mu(\xi)$ is L^p-improving whenever $\xi \in (0, 1/2)$.

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Reçu par la Rédaction le 21. 10. 1981;
en version modifiée le 18. 09. 1982