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FAMILIES OF INDEPENDENT SETS IN FINITE UNARY ALGEBRAS

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This paper contains remarks concerning E. Marczewski's problem on the set-theoretical characterization of families of independent sets in abstract algebras. Namely the case of finite unary algebras is solved here.

This completes J. S. Johnson's result [4] in which families of independent sets are characterized for mono-unary algebras, i.e., algebras with one unary fundamental operation.

Condition used here for characterization of families of independent sets in finite unary algebras are similar to the assumptions of the theorems of Świerczkowski and Fajtlowicz ([7], [2]), but there they were only sufficient.

K. Urbanik gave in [8] some necessary conditions. Other necessary conditions are given in [1] and [2].

For notions and symbols used in this paper see [5] and [6]. In particular, we call an element a *self-dependent* if $\{a\} \notin \boldsymbol{Ind}$, we denote by $C(X)$ the subalgebra generated by set X, and by $|X|$ the power of the set X.

Let $(A; \boldsymbol{J})$ be a pair in which A is an arbitrary set and $\boldsymbol{J}$ a family of subsets of A. An algebra $(A; \boldsymbol{F})$ is called an $(A; \boldsymbol{J})$-*algebra* if $\boldsymbol{Ind}(A; \boldsymbol{F}) = \boldsymbol{J}$.

We say that the *family* $\boldsymbol{J}$ *is of character* n if every set all subsets of which having at most n elements belong to $\boldsymbol{J}$ belongs to $\boldsymbol{J}$.

For every $(A; \boldsymbol{J})$ we put $U = U(\boldsymbol{J}) = \{x : \{x\} \in \boldsymbol{J}\}$, and we denote by $\varrho = \varrho(\boldsymbol{J})$ a relation on the set U such that $x\varrho y$ if and only if $x = y$ or $\{x, y\} \notin \boldsymbol{J}$.

By a *graph* we mean here a relational structure $(X; \sigma, \bar{\sigma})$, where σ is a binary symmetric and reflexive relation, and $x\bar{\sigma}y \Leftrightarrow \sim x\sigma y$. We introduce the relation $\bar{\sigma}$ together with σ because we shall study mappings h of graphs such that $x\sigma y \Leftrightarrow h(x)\sigma h(y)$. Such mappings h are the only homomorphisms of graphs in our sense.

Let $\mathscr{G} = \mathscr{G}(\boldsymbol{J}) = \big(U(\boldsymbol{J});\ \varrho(\boldsymbol{J}), \bar{\varrho}(\boldsymbol{J})\big)$.

Let B be an arbitrary set. By $\mathscr{H}(B, k)$ we shall denote the graph $(X; \sigma, \bar{\sigma})$ in which X *is the set of all* k-*element subsets of* B and the relation σ holds if and only if $x \cap y \neq \emptyset$, and $\bar{\sigma}$ is the complement of σ.

By $\gamma(\mathscr{G})$ we shall denote the power of the smallest set B such that for some natural k there exists a homomorphism $h\colon \mathscr{G} \to \mathscr{H}(B; k)$.

Let us prove the

LEMMA. *If $\mathfrak{A}$ is a finite unary algebra in which every self-dependent element is an algebraic constant, then the relation $\varrho = \varrho(\mathbf{Ind}\,\mathfrak{A})$ is an equivalence in which every class of equivalent elements has the same power.*

Indeed the relation $x\varrho y$ holds if and only if $\{x\}, \{y\} \in \mathbf{Ind}$ and $x \in C\{y\}$. The symmetry of this relation follows from the finiteness of A and the independence of x and y.

Note that (see Johnson [4])

(F) *The family of independent sets in unary algebras is of character* 2.

Let us incidentally remark that, more generally,

(F′) *If in an algebra $\mathfrak{A}$ every algebraic operation depends on at most n variables, then $\mathbf{Ind}\,\mathfrak{A}$ is of character $2n$.*

We also have

(F″) *If $\mathfrak{A}$ is a unary finite algebra and X and Y are disjoint independent sets, then $X \cup Y \in \mathbf{Ind}\,\mathfrak{A}$ if and only if $C(X) \cap C(Y) = C(\emptyset)$.*

In other words, (F″) implies that unary algebras have the property JIS [3].

From now on A denotes a finite non-empty set.

THEOREM. *For the existence of a finite unary $(A; \mathbf{J})$-algebra it is necessary and sufficient that $\mathbf{J}$ be of character* 2 *and that we have*

(i) *$\varrho(\mathbf{J})$ is an equivalence in which every class of equivalent elements has the same power*

or

(ii) $\gamma\big(\mathscr{G}(\mathbf{J})\big) \leqslant |A \setminus U(\mathbf{J})|$.

Proof. By (F) the existence of an $(A; \mathbf{J})$-algebra implies that $\mathbf{J}$ is of character 2.

Let $\mathfrak{A} = (A; \mathbf{F})$ be a finite unary algebra, $\varrho = \varrho(\mathbf{Ind}\,\mathfrak{A})$ and $U = U(\mathbf{Ind}\mathfrak{A})$. Let us suppose that ϱ does not satisfy (i). Then in the algebra $\mathfrak{A}$ there exists a non-constant algebraic operation f and an element $a \in U$ such that $f(a) \in A \setminus U$. Indeed, in the other case the set $U \cup C(\emptyset)$ would be a subalgebra satisfying the condition of the Lemma and the conclusion of this Lemma gives again (i).

Thus let $a \in U$ and $f(a) \in A \setminus U$. Since $\{a\} \in \mathbf{Ind}\mathfrak{A}$, we have $f(a) \notin C(\emptyset)$. For $x \in U$ let $h(x)$ denote the set $C\{x\} \cap \big(A \setminus \big(U \cup C(\emptyset)\big)\big)$. From what we said it follows that $f(a) \in h(a)$. Since $a \in h(x)$ if and only if a is self-dependent but is not an algebraic constant and since every two elements of U generate isomorphic subalgebras, we have $|h(x)| = |h(y)|$ for every $x, y \in U$.

Using (F″) when X and Y are singletons, we conclude that h is a homomorphism of the graph $\mathscr{G}(\mathbf{Ind}\mathfrak{A})$ into the graph $\mathscr{H}\big(A \setminus U; |h(x)|\big)$.

Thus the necessity is proved.

Now let us suppose that the pair $(A; \boldsymbol{J})$ satisfies (i). Let us order each equivalence class of ϱ into a sequence $x_1, \ldots, x_n$, and put $f(x_k) = x_{k+1}$ for $k < n$, $f(x_n) = x_1$ and $f(x) = x$ for $x \in A \setminus U(\boldsymbol{J})$. It is easy to see that an algebra $(A; f, c)_{c \in A \setminus U}$ is an $(A; \boldsymbol{J})$-algebra.

If the pair $(A; \boldsymbol{J})$ satisfies (ii), then there exists a homomorphism of the graph $\big(U; \varrho(\boldsymbol{J}), \bar{\varrho}(\boldsymbol{J})\big)$ into the graph $\mathscr{H}(A \setminus U, k)$ for some natural k. For every element $a \in U(\boldsymbol{J})$ let us order the elements of the set $h(a)$ and let us put $f_i(x)$ equal to the i-th element of $h(x)$ for $x \in U$ and $f_i(x) = x$ for $x \in A \setminus U$.

We can assume that the pair $(A; \boldsymbol{J})$ does not satisfy (i). In this case, in the algebra $\mathfrak{A} = (A; f_i)_{i=1,\ldots,|h(x)|}$ there are no constants. Thus from condition (F″) and the fact that h is a homomorphism it follows at once that the algebra $\mathfrak{A}$ is an $(A; \boldsymbol{J})$-algebra.

Thus the sufficiency is proved as well.

Remark 1. From the last part of the proof it follows that if $\mathfrak{A} = (A; \boldsymbol{F})$ is a unary finite algebra with ***Ind*** failing to satisfy (i), then there exists a unary algebra $\mathfrak{B} = (A; \boldsymbol{G})$ such that ***Ind*** $\mathfrak{A}$ = ***Ind*** $\mathfrak{B}$ and there are no constants in $\mathfrak{B}$. To have this, it is essential to assume that ***Ind*** does not satify (i) as we can see taking the algebra $(A; c)$ in which $A = \{a, b, c\}$ and ***Ind*** $= \big\{\{a\}, \{b\}, \{a, b\}\big\}$.

Remark 2. To conclude that each of conditions (i) and (ii) is sufficient for the existence of an $(A; \boldsymbol{J})$-algebra it is not necessary to require the finiteness of A. Indeed, we can see this by a slight modification of our proof.

REFERENCES

[1] S. Fajtlowicz, *A remark of independence in projective spaces*, Colloquium Mathematicum 19 (1968), p. 23-25.

[2] — *Properties of the family of independent subsets of a general algebras*, ibidem 14 (1966), p. 225-231.

[3] — and K. Głazek, *Independence in separable variables algebra*, ibidem 18 (1967), p. 221-224.

[4] J. S. Johnson, *Marczewski independence in mono-unary algebras*, this fascicle, p. 7-11.

[5] E. Marczewski, *Independence and homomorphisms in abstract algebras*, Fundamenta Mathematicae 49 (1960), p. 45-61.

[6] — *Independence in abstract algebras*, Colloquium Mathematicum 14 (1966), p. 169-188.

[7] S. Świerczkowski, *A sufficient condition for independence*, ibidem 9 (1962), p. 39-42.

[8] K. Urbanik, *Remarks on independence in finite algebras*, ibidem 11 (1963), p. 1-12.

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